

The Mathematics Education
Volume-LX, No. 1, March 2026

ISSN 0047-6269

Journal website: <https://internationaljournalsiwan.com/Mathematic.php>

ORCID Link: <https://orcid.org/0009-0006-7467-6080>

International Impact Factor: 7.75 <https://impactfactorservice.com/home/journal/2295>

Google Scholar: <https://scholar.google.com/citations?hl=en&user=UOfM8B4AAAAJ>

Refereed and Peer-Reviewed Quarterly Journal



Fixed Point Theorems for Fuzzy Contraction Mappings in A Dislocated 3-Points b -Metric Spaces

by **Vaishali Prajapati**, *Research Scholar,*
Department of Mathematics,
Maharaja Chhatrasal Bundelkhand University,
Chhatarpur-471001, India

N.P. Prajapati, *Department of Mathematics,*
Maharaja Chhatrasal Bundelkhand University,
Chhatarpur-471001, India

(Received: February 9, 2026; Accepted: February 23, 2026; Published Online: March 31, 2026)

Abstract:

A very impressive research work has been devoted in the last two decades to obtaining fixed point theorems in b -metric space. The purpose of this paper is to obtain some rational and common fixed point results for fuzzy mapping in a complete dislocated 3-points b -metric spaces with Hausdorff b -metric spaces.

Keywords: Dislocated metric spaces, Fuzzy fixed point, Dislocated Hausdorff b -metric spaces, common fixed point, dislocated 3-points b -metric spaces, rational type contraction

Introduction:

The concept of fixed point is fundamental in mathematics and its applications. A fixed point of mapping $T: X \rightarrow X$ is a point $x \in X$ such that $T(x) = x$. Which establish the existence and uniqueness of such points under specific conditions Fixed point theorems for contraction mappings is one of the central results of solving the large number of applications in mathematical analysis. In 1906, Fréchet [9] introduced metric space. Later, in 1922 Banach [3] introduced Banach contraction principle is one of the remarkably results which has played a crucial role in the evolution of metric fixed point theory.

This principle and its variants provided a useful equipment in guaranteeing the existence and uniqueness of solutions of various nonlinear problems; differential equations, integral equations, optimization problems, variational inequalities. A host of this principle has been made modified and extended by several mathematicians in different perspectives, some of them are as follows. In 1965, L.A. Zadeh [27] introduced the concept of fuzzy sets which become active field of research for researchers. Czerwik [7] had an excellent idea to introduce the concepts of b -metric spaces by weakening the coefficient of the triangle inequality and generalized Banach's contraction principle. Hussain et al. [12] introduced the new concept of dislocated b -metric space as a generalization of metric space and established to prove some common fixed-point results for four mappings satisfying the generalized weak contractive conditions in partially ordered dislocated b -metric space.

In 1975, Dass and Gupta [8] rational type contraction, if \exists a number $\alpha, \beta \in [0, 1)$ with $\alpha + \beta < 1$ such that

$$d(Tx, Ty) \leq \alpha \frac{d(y, Ty)[1 + d(x, Tx)]}{1 + d(x, y)} + \beta d(x, y)$$

condition to prove a generalization of Banach's fixed point.

Recently years various applications of fixed-point theorems have challenged researchers to introduce different generalization of metric spaces, like Ciric Type rational fuzzy F -contraction, fuzzy contraction in b -metric spaces, b -metric spaces, S -metric spaces, rectangular S -metric spaces, dislocated quasi metric spaces and dislocated quasi b -metric spaces, (see [6, 13, 16, 17, 19, 20, 22, 25]).

In view of the above considerations, we prove some fixed-point results for fuzzy contraction mapping with rational expression in generalization dislocated b -metric spaces.

Preliminaries:

Definition 1.1: [7] Let X be a nonempty set and $s \geq 1$ be a given real number. If a function $d : X \times X \rightarrow R^+$ satisfies the following conditions;

- i. $d(x, y) = 0$ iff $x = y$
- ii. $d(x, y) = d(y, x)$
- iii. $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y \in X$

Then the pair (X, d) is called a b -metric space.

Definition 1.2: [21] Let X be a nonempty set. A function $d : X \times X \rightarrow R^+$. A pair (X, d) is called a distance space. If d satisfies the following conditions,

- i. $d(x, y) = 0$ iff $x = y$
- ii. $d(x, y) = d(y, x)$
- iii. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$

then, a function $d : X \times X \rightarrow R^+$ is called a dislocated metric on X . If d is a dislocated metric on X , then the pair (X, d) is called to be a dislocated metric space.

Definition 1.3: [10, 20] A fuzzy set in X is a function with domain X and value in $[0, 1]$, $F(X)$ is the collection of all fuzzy set in X . If A is a fuzzy set and $x \in X$ then the function value $A(x)$ is called the grade of membership of x in A . The α -level set of fuzzy set A , is denoted by $[A]_\alpha$, and defined as;

$$[A]_\alpha = \{x : A(x) \geq \alpha\} \text{ where } \alpha \in (0, 1] \quad [A]_0 = \overline{\{x : A(x) > 0\}}$$

Definition 1.4: [5, 2] Let X be any nonempty set and Y be a metric space. A mapping T is called a fuzzy mapping from X into $F(Y)$. A fuzzy mapping T is a fuzzy subset on $X \times Y$ with membership of $T(x)(y)$. The function $T(x)(y)$ is the grade of membership of y in $T(x)$. For convenience, we denote the α -level set of $T(x)$ by $[T(x)]_\alpha$ instead of $[T(x)]_\alpha$.

4 The Mathematics Education [Vol. LX (1), March 26]

Definition 1.5: [18, 22] A point $x \in X$ is called a fuzzy fixed point of a fuzzy mapping $T : X \rightarrow F(X)$ if there exists $\alpha \in (0, 1]$ such that $x \in [Tx]_\alpha$.

Definition 1.6: [11, 4] Let X be a nonempty set. A function $d_{lb} : X \times X \rightarrow [0, \infty)$ is called dislocated b -metric (or simply d_{lb} -metric) if, for any $x, y, z \in X$ and $b \geq 1$, the following conditions hold;

- i. If $d_{lb}(x, y) = 0$, then $x = y$
- ii. $d_{lb}(x, y) = d_{lb}(y, x)$
- iii. $d_{lb}(x, y) \leq b[d_{lb}(x, z) + d_{lb}(z, y)]$

The pair (X, d_{lb}) is called a dislocated b -metric spaces. It should be noted that the class of d_{lb} metric spaces is effectively larger than that of d_l metric spaces, since d_{lb} is a d_l metric when $b = 1$. It is clear that if $d_{lb}(x, y) = 0$, then from (i), $x = y$. But if $y = (X, d_{lb})$, may not be 0.

Example: [21] If $X = R^+ \cup \{0\}$ then $d_{lb}(x, y) = (x + y)^2$ defines a dislocated b -metric d_{lb} on X .

Definition 1.7: [1] Let (X, d_{lb}) be a dislocated b -metric space.

- i. A sequence $\{x_n\}$ in (X, d_{lb}) is called Cauchy sequence if, given $\epsilon > 0$, there corresponds $n_0 \in N$ such that, for all $n, m \geq n_0$, we have $d_{lb}(x_m, x_n) < \epsilon$ or $\lim_{n, m \rightarrow \infty} d_{lb}(x_m, x_n) = 0$.
- ii. A sequence $\{x_n\}$ dislocated b -converges (for short d_{lb} -converges) to x if $\lim_{n, m \rightarrow \infty} d_{lb}(x_n, n) = 0$. In this case x is called a d_{lb} limit of $\{x_n\}$.

Definition 1.8: [24] Let K be a nonempty subset of dislocated b -metric space X , and let $x \in X$. An element $y_0 \in K$ is called a best approximation in K if

$$d_{lb}(x, K) = d_{lb}(x, y_0),$$

where $d_{lb}(x, K) = \inf_{y \in K} d_{lb}(x, y)$.

If each $x \in X$ has at least one best approximation in K , then K is called a proximal set. Denoted by $P(X)$ the set of all proximal subset of X .

Definition 1.9: [24] The function $H_{d_{lb}} : P(X) \times P(X) \rightarrow R^+$, defined by

$$H_{d_{lb}}(A, B) = \max \{ \sup_{a \in A} d_{lb}(a, B), \sup_{b \in B} d_{lb}(A, b) \},$$

is called dislocated Hausdorff b -metric on $P(X)$.

Lemma 1.10: [24] Let A and B be nonempty proximal subset of a dislocated b -metric space (X, d_{lb}) . If $a \in A$ then

$$d(a, B) \leq H(A, B).$$

Lemma 1.11: [24] Let (X, d_{lb}) be a dislocated metric space. Let $(P(X), H_{d_{lb}})$ be a dislocated Hausdorff b -metric space. Then for all $A, B \in P(X)$ and for each $a \in A$ there exists $b_a \in B$ satisfying

$$d_{lb}(A, b) = d_{lb}(a, b_a)$$

Then $H_{d_{lb}}(A, B) \geq d_{lb}(a, b_a)$.

Definition 1.12: [14, 19] Let (X, d) be a complete metric space and $S, T : X \rightarrow X$ be two given mappings. Then we say that $x \in X$ is common fixed point of S and T if $x = Sx = Tx$.

Main Results:

In this section, we begin with the following theorem and definition.

Definition 1.13: Let X be a non empty set. A function $d_{lb} : X \times X \times X \rightarrow [0, \infty)$ is a dislocated 3-point b -metric if there exists $b \geq 1$ such that for all $x, y, z, w \in X$ the following condition hold

- i. Dislocation: $d_{lb}(x, x, x) \geq 0$ and $d_{lb}(x, y, z) = 0 \Leftrightarrow x = y = z$
- ii. Symmetry: $d_{lb}(x, y, z) = d_{lb}(y, z, x) = d_{lb}(z, x, y)$
- iii. Ternary triangle Inequality: $d_{lb}(x, y, z) \leq b[d_{lb}(x, y, w) + d_{lb}(x, w, z) + d_{lb}(w, y, z)]$. Then (X, d_{lb}) is called a dislocated 3-point b -metric space.

Theorem 1: Let (X, d_{lb}) be a complete dislocated 3-point b -metric space with constant $b \geq 1$. Let $T : X \rightarrow F(X)$ be a fuzzy mapping. Suppose there exists $\alpha(x) \in (0, 1]$, satisfying the following condition

$$H_{d_{lb}}([Tx]_{\alpha(x)}, [Ty]_{\alpha(y)}, [Tz]_{\alpha(z)}) \leq a_1 d_{lb}(y, y, [Ty]_{\alpha(y)}) \frac{[1 + d_{lb}(x, x, [Tx]_{\alpha(x)})]}{1 + d_{lb}(x, y, z)} + a_2 d_{lb}(z, [Ty]_{\alpha(y)}, [Tz]_{\alpha(z)}) + a_3 d_{lb}(x, y, z) \quad (1)$$

For all $x, y, z \in X$; $x \neq y \neq z$ and $a_1 + a_2 + a_3 < 1$ where $a_1, a_2, a_3 \geq 0$, then there exists $x^* \in [Tx^*]_{\alpha(x^*)}$.

Proof: Let $x_0 \in X$ be an arbitrary point in X . Let $x_1 \in [Tx_0]_{\alpha(x_0)}$ be an element such that $d_{lb}(x_0, x_0, [Tx_0]_{\alpha(x_0)}) = d_{lb}(x_0, x_0, x_1)$. Again let $x_2 \in [Tx_1]_{\alpha(x_1)}$ be an element such that $d_{lb}(x_1, x_1, [Tx_1]_{\alpha(x_1)}) = d_{lb}(x_1, x_1, x_2)$. Continuing in this process, we construct a sequence x_n of points in X , such that $x_n \in [Tx_{n-1}]_{\alpha(x_{n-1})}$ for all $n \in N$. Now by using lemma (1.11) and equation (1) we get,

$$\begin{aligned} d_{lb}(x_n, x_n, x_{n+1}) &\leq H_{d_{lb}}([Tx_{n-1}]_{\alpha(x_{n-1})}, [Tx_{n-1}]_{\alpha(x_{n-1})}, [Tx_n]_{\alpha(x_n)}) \\ &\leq \frac{a_1 d_{lb}(x_{n-1}, x_{n-1}, x_n) [1 + d_{lb}(x_{n-1}, x_{n-1}, x_n)]}{1 + d_{lb}(x_{n-1}, x_{n-1}, x_n)} + a_2 d_{lb}(x_n, x_n, x_{n+1}) \\ &\quad + a_3 d_{lb}(x_{n-1}, x_{n-1}, x_n) d_{lb}(x_n, x_n, x_{n+1}) \\ &\leq a_1 d_{lb}(x_{n-1}, x_{n-1}, x_n) + a_2 d_{lb}(x_n, x_n, x_{n+1}) + a_3 d_{lb}(x_{n-1}, x_{n-1}, x_n) \\ &\leq \left(\frac{a_1 + a_3}{1 - a_2} \right) d_{lb}(x_{n-1}, x_{n-1}, x_n) \end{aligned} \quad (2)$$

Let $\rho = \left(\frac{a_1 + a_3}{1 - a_2} \right) < 1$

$$\begin{aligned} d_{lb}(x_n, x_n, x_{n+1}) &\leq \rho d_{lb}(x_{n-1}, x_{n-1}, x_n) \\ &\leq \rho^2 d_{lb}(x_{n-2}, x_{n-2}, x_{n-1}) \\ &\quad \vdots \\ &\leq \rho^n d_{lb}(x_0, x_0, x_1) \end{aligned} \quad (3)$$

For all $n \in N$ and for any $n > m$, where $n, m \in N$. We have by using the condition (iii) condition of dislocated 3-points b -metric spaces.

$$\begin{aligned}
 d_{lb}(x_m, x_m, x_n) &\leq b d_{lb}(x_n, x_n, x_{n+1}) + b^2 d_{lb}(x_{m+1}, x_{m+1}, x_{m+2}) + \dots \\
 &\quad + b^{n-m} d_{lb}(x_{n-1}, x_{n-1}, x_n) \\
 &\leq b \rho^m d_{lb}(x_0, x_0, x_1) + b^2 \rho^{m+1} d_{lb}(x_0, x_0, x_1) + \dots \\
 &\quad + b^{n-m} \rho^{n-1} d_{lb}(x_0, x_0, x_1) \\
 &\leq b \rho^m [1 + b\rho + b^2 \rho^2 + \dots + b^{n-m-1} \rho^{n-m-1}] d_{lb}(x_0, x_0, x_1) \\
 &\leq b \rho^m [1 + b\rho + b^2 \rho^2 + \dots + b^{n-m-1} \rho^{n-m-1}] d_{lb}(x_0, x_0, x_1) \\
 &\leq \frac{b \rho^m}{1 - b\rho} d_{lb}(x_0, x_0, x_1) \tag{4}
 \end{aligned}$$

Taking the limit of $d_{lb}(x_m, x_m, x_n)$ as $n, m \rightarrow \infty$.

We have,

$$\lim_{n, m \rightarrow \infty} d_{lb}(x_m, x_m, x_n) = \lim_{n, m \rightarrow \infty} \frac{b \rho^m}{1 - b\rho} d_{lb}(x_0, x_0, x_1) = 0$$

This proves that $\{x_n\}$ is Cauchy sequence in X .

Since X is complete, $\exists x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Now by lemma (1.11) and equation (1) we have

$$\begin{aligned}
 d_{lb}(x^*, x^*, [Tx^*]_{\alpha(x^*)}) &\leq b \left[d_{lb}(x^*, x^*, x_{n+1}) + d_{lb}(x^*, x^*, [Tx^*]_{\alpha(x^*)}) \right. \\
 &\quad \left. + d_{lb}(x^*, x^*, [Tx^*]_{\alpha(x^*)}) \right] \\
 &\leq b \left[d_{lb}(x^*, x^*, x_{n+1}) + d_{lb}(x^*, x_{n+1}, [Tx^*]_{\alpha(x^*)}) \right. \\
 &\quad \left. + Hd_{lb}([Tx_n]_{\alpha(x_n)}, [Tx^*]_{\alpha(x^*)}, [Tx^*]_{\alpha(x^*)}) \right] \\
 &\leq b \left[d_{lb}(x^*, x^*, x_{n+1}) + d_{lb}(x^*, x_{n+1}, [Tx^*]_{\alpha(x^*)}) \right. \\
 &\quad + a_1 d_{lb}(x^*, x^*, [Tx^*]_{\alpha(x^*)}) \frac{1 + d_{lb}(x_n, x_n, [Tx_n]_{\alpha(x_n)})}{1 + d_{lb}(x_n, x^*, x^*)} \\
 &\quad \left. + a_2 d_{lb}(x^*, [Tx^*]_{\alpha(x^*)}, [Tx^*]_{\alpha(x^*)}) + a_3 d_{lb}(x_n, x^*, x^*) \right] \tag{5}
 \end{aligned}$$

On taking $n \rightarrow \infty$ in (5), we get

$$d_{lb}(x^*, x^*, [Tx^*]_{\alpha(x^*)}) \leq b \left[a_1 d_{lb}(x^*, x^*, [Tx^*]_{\alpha(x^*)}) + a_2 d_{lb}(x^*, x^*, [Tx^*]_{\alpha(x^*)}) \right. \\ \left. (1 - (a_1 + a_2)) d_{lb}(x^*, x^*, [Tx^*]_{\alpha(x^*)}) \right] \leq 0 \quad (6)$$

From (6), we get

$$x^* \in [Tx^*]_{\alpha(x^*)}$$

Hence, $x^* \in X$ is the fixed point.

Theorem 2: Let (X, d_{lb}) be a complete dislocated 3-point b -metric space with constant $b \geq 1$. Let $T : X \rightarrow F(X)$ be a fuzzy mapping. Suppose there exists $\alpha_{S(x)}, \alpha_{T(x)} \in (0, 1]$ satisfying the following conditions:

$$H_{d_{lb}}([Tx]_{\alpha T(x)}, [Ty]_{\alpha T(y)}, [Sz]_{\alpha S(z)}) \\ \leq \frac{a_1 d_{lb}(x, x, [Tx]_{\alpha T(x)}) [1 + d_{lb}(y, y, [Ty]_{\alpha T(y)})]}{1 + d_{lb}(x, y, z)} \\ + a_2 d_{lb}(z, [Ty]_{\alpha T(y)}, [Sz]_{\alpha S(z)}) + a_3 d_{lb}(x, y, z) \quad (1)$$

For all $x, y, z \in X; x \neq y \neq z$ and $a_1, a_2, a_3 \geq 0$ and $a_1 + a_2 + a_3 \leq 1$, then there exists $x^* \in X$ such that x^* is common fixed point of S and T .

Proof: Let $x_0 \in X$ be an arbitrary point in X . Let $x_1 \in [Tx_0]_{\alpha T(x_0)}$ be an element such that $d_{lb}(x_0, x_0, [Tx_0]_{\alpha T(x_0)}) = d_{lb}(x_0, x_0, x_1)$. Again let $x_2 \in [Sx_1]_{\alpha S(x_1)}$ be an element such that $d_{lb}(x_1, x_1, [Sx_1]_{\alpha S(x_1)}) = d_{lb}(x_1, x_1, x_2)$. Continuing in this process, we construct a sequence x_n of points in X , such that $x_{2n+1} \in [Tx_{2n}]_{\alpha T(x_{2n})}$ and $x_{2n+2} \in [Sx_{2n+1}]_{\alpha S(x_{2n+1})}$ for all $n \in N$. Now by using lemma (1.12) and equation (1) we get,

$$d_{lb}(x_{2n+1}, x_{n+1}, x_{2n+2}) \leq H_{d_{lb}}([Tx_{2n}]_{\alpha T(x_{2n})}, [Tx_{2n}]_{\alpha T(x_{2n})}, [Sx_{2n+1}]_{\alpha S(x_{2n+1})}) \\ \leq \frac{a_1 d_{lb}(x_{2n}, x_{2n}, [Tx_{2n}]_{\alpha T(x_{2n})}) [1 + d_{lb}(x_{2n}, x_{2n}, [Tx_{2n}]_{\alpha T(x_{2n})})]}{1 + d_{lb}(x_{2n}, x_{2n}, x_{2n+1})} \\ + a_2 d_{lb}(x_{2n+1}, x_{2n+1}, x_{2n+2}) + a_3 d_{lb}(x_{2n}, x_{2n}, x_{2n+1})$$

$$d_{lb}(x_{2n+1}, x_{2n+1}, x_{2n+2}) [1 - a_2] \leq (a_1 + a_3) d_{lb}(x_{2n}, x_{2n}, x_{2n+1})$$

$$d_{lb}(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq \frac{(a_1 + a_3)}{(1 - a_2)} d_{lb}(x_{2n}, x_{2n}, x_{2n+1}) \quad (2)$$

Similarly, by using condition (2) of definition of dislocated 3-points b -metric spaces, we have

$$d_{lb}(x_{2n+2}, x_{2n+1}, x_{2n+1}) \leq H_{d_{lb}}([Sx_{2n+1}]_{\alpha S(x_{2n+1})}, [Tx_{2n}]_{\alpha T(x_{2n})}, [Tx_{2n}]_{\alpha T(x_{2n})})$$

$$\leq \frac{a_1 d_{lb}(x_{2n}, x_{2n}, [Tx_{2n}]_{\alpha T(x_{2n})}) [1 + d_{lb}(x_{2n+1}, x_{2n+1}, [Sx_{2n+1}]_{\alpha S(x_{2n+1})})]}{1 + d_{lb}(x_{2n+1}, x_{2n+1}, x_{2n})}$$

$$+ a_2 d_{lb}(x_{2n}, x_{2n+1}, x_{2n+1}) + a_3 d_{lb}(x_{2n+1}, x_{2n}, x_{2n})$$

$$\leq a_1 d_{lb}(x_{2n+1}, x_{2n+1}, x_{2n+2}) + a_2 d_{lb}(x_{2n}, x_{2n+1}, x_{2n+1}) + a_3 d_{lb}(x_{2n+1}, x_{2n}, x_{2n})$$

$$d_{lb}(x_{2n+2}, x_{2n+1}, x_{2n+1}) [1 - a_1] \leq (a_2 + a_3) d_{lb}(x_{2n}, x_{2n}, x_{2n+1})$$

$$d_{lb}(x_{2n+2}, x_{2n+1}, x_{2n+1}) \leq \frac{(a_2 + a_3)}{(1 - a_1)} d_{lb}(x_{2n}, x_{2n}, x_{2n+1}) \quad (3)$$

From (2) and (3),

$$d_{lb}(x_{2n+2}, x_{2n+1}, x_{2n+1}) \leq \frac{(a_1 + a_2 + 2a_3)}{(2 - (a_1 + a_2))} d_{lb}(x_{2n}, x_{2n}, x_{2n+1}) \quad (4)$$

$$\text{Let } \mu = \frac{(a_1 + a_2 + 2a_3)}{(2 - (a_1 + a_2))} \leq 1$$

then by (4), we get

$$d_{lb}(x_{2n+2}, x_{2n+1}, x_{2n+1}) \leq \mu d_{lb}(x_{2n}, x_{2n}, x_{2n+1}) \quad (5)$$

For all $n \in N$. Repeated application of (5), we have

$$d_{lb}(x_{2n+2}, x_{2n+1}, x_{2n+1}) \leq \mu d_{lb}(x_{2n}, x_{2n}, x_{2n+1})$$

$$\leq \mu^2 d_{lb}(x_{2n-1}, x_{2n-1}, x_{2n})$$

$$\begin{aligned}
&\leq \mu^3 d_{lb}(x_{2n-2}, x_{2n-2}, x_{2n-1}) \\
&\quad \vdots \\
&\quad \vdots \\
&\leq \mu^{n+1} d_{lb}(x_0, x_0, x_1)
\end{aligned} \tag{6}$$

For all $n \in N$. Again, for any $n > m$, where $n, m \in N$, we have by using the condition (iii) of definition of (dislocated 3-points b -metric spaces).

$$\begin{aligned}
d_{lb}(x_m, x_m, x_n) &\leq b(d_{lb}(x_n, x_n, x_{n+1}) + b^2 d_{lb}(x_{m+1}, x_{m+1}, x_{m+2}) + \dots \\
&\quad + b^{n-m} d_{lb}(x_{n-1}, x_{n-1}, x_n)) \\
&\leq b \mu^m d_{lb}(x_0, x_0, x_1) + b^2 \mu^{m+1} d_{lb}(x_0, x_0, x_1) + \dots \\
&\quad + b^{n-m} \mu^{n-1} d_{lb}(x_0, x_0, x_1) \\
&\leq b \mu^m [1 + b\mu + b^2 \mu^2 + \dots + b^{n-m-1} \mu^{n-m-1}] d_{lb}(x_0, x_0, x_1) \\
&\leq \frac{b \mu^m}{1 - b\mu} d_{lb}(x_0, x_0, x_1)
\end{aligned} \tag{7}$$

Taking limit of $d_{lb}(x_m, x_m, x_n)$ as $m, n \rightarrow \infty$, we have,

$$\lim_{n, m \rightarrow \infty} d_{lb}(x_m, x_m, x_n) = \lim_{n, m \rightarrow \infty} \frac{b \mu^m}{1 - b\mu} d_{lb}(x_0, x_0, x_1) = 0$$

This proves that $\{x_n\}$ is Cauchy sequence in X .

Since X is complete, there exists $x^* \in X$ such that $x_n \rightarrow x^*$ as $n \rightarrow \infty$. Now by lemma (1.11) and equation (1) we now prove that $x^* \in X$ have a common fixed point of S and T .

$$\begin{aligned}
d_{lb}(x^*, x^*, [Sx^*]_{\alpha S(x^*)}) &\leq b \left[d_{lb}(x^*, x^*, x_{2n+1}) + d_{lb}(x^*, x_{2n+1}, [Sx^*]_{\alpha S(x^*)}) \right. \\
&\quad \left. + d_{lb}(x_{2n+1}, x^*, [Sx^*]_{\alpha S(x^*)}) \right] \\
&\leq b \left[d_{lb}(x^*, x^*, x_{2n+1}) + d_{lb}(x^*, x_{2n+1}, [Sx^*]_{\alpha S(x^*)}) \right. \\
&\quad \left. + H d_{lb}([Sx_{2n+2}]_{\alpha S(x_{2n+2})}, [Sx^*]_{\alpha S(x^*)}, [Sx^*]_{\alpha S(x^*)}) \right]
\end{aligned}$$

$$\begin{aligned} &\leq b \left[d_{lb}(x^*, x^*, x_{2n+1}) + d_{lb}(x^*, x_{2n+1}, [Sx^*]_{\alpha S(x^*)}) \right. \\ &\quad + a_1 d_{lb}(x^*, x^*, [Sx^*]_{\alpha S(x^*)}) \\ &\quad \frac{1 + d_{lb}(x_{2n+2}, x_{2n+2}, [Sx_{2n+2}]_{\alpha S(x_{2n+2})})}{1 + d_{lb}(x_{2n+2}, x^*, x^*)} \\ &\quad + a_2 d_{lb}(x^*, [Sx^*]_{\alpha S(x^*)}, [Sx^*]_{\alpha S(x^*)}) \\ &\quad \left. + a_3 d_{lb}(x_{2n+2}, x^*, x^*) \right] \end{aligned} \tag{8}$$

On taking $n \rightarrow \infty$ in (8), we get

$$\begin{aligned} d_{lb}(x^*, x^*, [Sx^*]_{\alpha S(x^*)}) &\leq b \left[a_1 d_{lb}(x^*, x^*, [Sx^*]_{\alpha S(x^*)}) \right. \\ &\quad \left. + a_2 d_{lb}(x^*, x^*, [Sx^*]_{\alpha S(x^*)}) \right] \\ d_{lb}(x^*, x^*, [Sx^*]_{\alpha S(x^*)}) &\leq b(a_1 + a_2) d_{lb}(x^*, x^*, [Sx^*]_{\alpha S(x^*)}) \\ [1 - (a_1 + a_2)] d_{lb}(x^*, x^*, [Sx^*]_{\alpha S(x^*)}) &\leq 0 \end{aligned} \tag{9}$$

From (9), we get

$$x^* \in [Sx^*]_{\alpha S(x^*)}$$

Hence, $x^* \in X$ is the fixed point of S .

Similarly, we can prove that $x^* \in X$ is the fixed point of T .

Hence, $x^* \in X$, have a common fixed point of S and T .

References:

1. Arshad, M., Shoaib, A., & Vetro, P. (2013) : Common fixed points of a pair of Hardy Rogers type mappings on a closed ball in ordered dislocated metric spaces. *Journal of Function Spaces*, 2013(1), 638181.
2. Azam, A. (2011) : Fuzzy fixed points of fuzzy mappings via a rational inequality. *Hacettepe Journal of Mathematics and Statistics*, 40(3), 421-431.

12 **The Mathematics Education [Vol. LX (1), March 26]**

3. Banach, S. (1922) : Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta mathematicae*, Vol. 3, No. 1, pp. 133-181.
4. Beg, I., Arshad, M., & Shoaib, A. (2015) : Fixed point in a closed ball in ordered dislocated quasi metric space. *Fixed Point Theory*, 16(2), 195-206.
5. Butnariu, D. (1982) : Fixed points for fuzzy mappings. *Fuzzy sets and Systems*, 7(2), 191-207.
6. Ćirić, L.B. (1974) : A generalization of Banach's contraction principle, *Proceedings of the American Mathematical society*, 45(2), 267-273.
<https://doi.org/10.1090/S0002-9939-1974-0356011-2>
7. Czerwik, S. (1993) : Contraction mappings in b -metric spaces. *Acta mathematica et informatica universitatis ostraviensis*, Vol. 1, No. 1, pp. 5-11.
8. Dass, Bal Khishan, and Satya Gupta : "An extension of Banach contraction principle through rational expression." *Indian J. Pure Appl. Math* 6.12 (1975): 1455-1458.
9. Fréchet, M. (1906) : Sur quelques points du calcul fonctionnel. *Rendiconti del circolo Matematico di Palermo*, Vol. 22, December, pp. 1-72.
10. Heilpern, S. (1981) : Fuzzy mappings and fixed-point theorem. *Journal of Mathematical Analysis and Applications*, 83(2), 566-569.
11. Hitzler, P., & Seda, A.K. (2000) : Dislocated topologies. *J. Electr. Eng*, 51(12), 3-7.
12. Hussain, N., Roshan, J.R., Parvaneh, V., & Abbas, M. (2013) : Common fixed-point results for weak contractive mappings in ordered b -dislocated metric spaces with applications. *Journal of inequalities and Applications*, Vol. 2013, No. 1, pp. 486.
13. Ibrahim, M.A., Raji, M., & Rauf, K. (2024) : Fuzzy metric spaces and corresponding fixed-point theorems for fuzzy type contraction. *Trans. Fuzzy Sets Syst*, 3(2), 51-65.

14. Kamalov, F., & Leung, H.H. (2023) : Fixed point theory : A review. *arXiv preprint arXiv: 2309.03226*.
15. Mittal, N., Kumar, P., & Ansari, Z.K. (2024) : Some fixed-point theorems under contraction conditions using linear and rational expression in dislocated metric space. *TWMS Journal of Applied and Engineering Mathematics*.
16. N.P. Prajapati, Vaishali Prajapati, Narendra Tiwari (2024) : Fixed point Theorems with Rational Expression in Rectangular S - metric spaces. *Unnati International Journal of Multidisciplinary Scientific Research*, Vol. 6, No. 7, pp. 75-80.
17. Pasicki, L. (2015) : Dislocated metric and fixed point theorems. *Fixed point theory and applications*, 2015(1), 82. DOI 10.1186/s13663-015-0328-z
18. Phiangsungnoen, S., & Kumam, P. (2015) : Fixed point theorems for multivalued fuzzy contractions in b -metric spaces. *J. Nonlinear Sci. Appl*, 8(1), 55-63.
19. Raji, M. (2023) : Generalized α - ψ contractive type mappings and related coincidence fixed point theorems with applications. *The Journal of Analysis*, 31(2), 1241-1256.
20. Rashid, M., Shahzad, A., & Azam, A. (2017) : Fixed point theorems for L -fuzzy mappings in quasi-pseudo metric spaces. *Journal of Intelligent & Fuzzy Systems*, 32(1), 499-507.
21. Ren, Yijie, Junlei Li, and Yanrong Yu. : "Common fixed-point theorems for nonlinear contractive mappings in dislocated metric spaces." *Abstract and Applied Analysis*. Vol. 2013, No. 1, Hindawi Publishing Corporation, 2013.
22. Shahzad, A., Shoaib, A., & Mahmood, Q. (2017) : Common fixed-point theorems for fuzzy mappings in b -metric space. *Italian Journal of Pure and Applied Mathematics*, 38, 419-427.

14 The Mathematics Education [Vol. LX (1), March 26]

23. Shahzad, A., Shoaib, A., Khammahawong, K., & Kumam, P. (2018, November) : New Ciric type rational fuzzy F -contraction for common fixed points. *In International Econometric Conference of Vietnam* (pp. 215-229). Cham: Springer International Publishing.
24. Shoaib, A., Kumam, P., Shahzad, A., Phiangsungnoen, S., & Mahmood, Q. (2018) : Fixed point results for fuzzy mappings in a b -metric space. *Fixed Point Theory and Applications*, 2018(1), 2.
25. Ur Rahman, M., Ali, A., Aziz, O., & Rahman, M.U. (2024) : Fixed Point Theorems in Dislocated Quasi Metric and Dislocated Quasi b -Metric Spaces. *Communications in Nonlinear Analysis*, 12(1), 1-10.
26. Weiss, M.D. (1975) : Fixed points, separation, and induced topologies for fuzzy sets. *Journal of Mathematical Analysis and Applications*, 50(1), 142-150.
27. Zadeh, L.A. (1965) : Fuzzy sets. *Information and control*, 8(3), 338-353.