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A certain studies on degree of approximation of functions by matrix transformation

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Abstract :

In this paper, we have established three new theorems which is are on certain studies on degree of approximation of functions by matrix transformation under very general conditions. This work is motivated by the works of Bojanczyk, A.W., Brent R.P., Hong F.R. and Sweet. D.R., "On the stability of the Burier and related Toepliz factorization algorithm", SIAM journal on Matrix Analysis and Applications, 16: 40-57, (1995). Obrechkoff, N, Formules asymptotiques pour les polynómes de Jacobi et surles series suivant les memespolynómes, Ann. Univ. 32, 39-135, (1936), Wilson and Bidwell E., A history of the theories from the age Descartes to the close of the nineteenth century, Bulletin of the American Mathematical Society, 26(4), 183-184, (1913).

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1. Introduction :

The study of regular, conservative and multiplicative matrices is important in the theory of summability for the first time, Lorentz [14] defined almost convergence of a sequences. Mazhar and Siddiqui [15] proved the important result i.e. a convergence sequence is almost convergent and the limits are the same. The almost summability methods are defined by the idea of almost convergence of a sequences. King [16] used the concept of almost convergence of a sequence introduced by Lorentz to define more general classes of matrices then those of regular and conservatives ones. Zygmund [17] has defined the approximation to a function by trigonometrical polynomials. It is familiar that every method fails to sum divergent series whose divergent series is too rapid and it, also fails to the sum the series whose divergence is too slow. The theorems which emody this principle belong to the class called "Tauberian theorems".

1.1 Definition :

If $a_{nk} = 0$ for $n > k$, A is called a lower semi-matrix or lower triangular matrix, if $a_{nk} = 0$ for $n < k$, A is called an upper semi-matrix or an upper triangular matrix. Further if $a_{nn} \neq 0$ for each n, A is said to be normal. The subject of infinite matrices, being a recent one, abounding in n good research problems. A very important application of matrices, namely to the theory of summability of divergent sequence and the series was initiated by Toeplitz [8, 9, 10, 11 and 13] in 1911. Since then, it has attached almost all researchers in the field of summability methods. In 2008, M.L. Mittle and V.N. Mishra [18] established a theorem on approximation of signals (functions) belonging to the Weighted $W(L_p, \xi(t))$, $(p \geq 1)$ - class by almost matrix summability method of its Fourier series. In 2014 L.N. Mishra, V.N. Mishra, K. Khatri, and Deepmala [19], established a theorem on the trigonometric approximation of signals belonging to generalized weighted Lipschitz $W(L^r, \xi(t))(r \geq 1)$ - class by matrix $(c^1. Np)$ operator of conjugate series of its Fourier series. Although, the concept of absolute summability was introduced as early as in 1911 by Fekete [2, 12] in case of Cesaro methods, and the same for Riesz and Abel method was defined by Obreachcoff [1, 3] and Whittaker [4] in 1928 and 1932 respectively for matrix transformation in general this was considered. In 1937 by Mears [7], Sunouchi [5] proof that in an RF-transformation.

QUOTE $\gamma_n = \sum_{k=1}^{l} J_{nk} u_k \gamma_n = \sum_{k=1}^{l} J_{nk} u_k$, in order that (1.1.1)

$$\sum_{k=1}^{l} |u_k| < \infty \quad \sum_{k=2}^{l} |\gamma_n - \gamma_{n-1}| < 1 \tag{1.1.2}$$

It is necessary and sufficient that

$$\sum_{n=2}^{l} | J_{nk} - J_{nk-1}| < M(G) \tag{1.1.3}$$

However, it was reviewed by Bosanquet [6], that for the RF- transformation (1.1) to exist, it is necessary that

$$| J_{nk}| < k_n(G) \tag{1.1.4}$$

holds; the constant $M(G)$ and $k_n(G)$ being independent of k. That is to say that the RF-transformation (1.1.1) is absolute convergence preserving if the transformation matrix $G = J_{nk}$ is B_A - matrix.

2. Preliminaries :

In this research paper, we proof the following theorems-

Theorem 2.1 : In RF-transformation $\gamma_n = \sum_{k=1}^{l} J_{nk} u_k$ necessary and sufficient condition for

$$\sum_{k=1}^{l} u_k = \lim_{n\to\infty} \gamma_n \text{ are that}$$

and

$$\sum_{k=2}^{l} |J_{nk} - J_{nk-1}| < M(G) \text{ Independent of } k; \tag{2.1.1}$$

$$|J_{nk}| < k_n(G) \text{ ; Independent of } k; \tag{2.1.2}$$

$$\lim_{n\to\infty} J_{nk} = 1, \ k = 1, 2... \tag{2.1.3}$$

Theorem 2.2 : Corresponding to every absolutely permanent FF-transformation matrix $P = (p_{nk})$, it is possible to construct an absolutely permanent RF-transformation matrix $G = (J_{nk})$ by defining,

$$J_{nk} = \sum_{j=k}^{l} p_{nj} \tag{2.2.1}$$

Moreover, the sequence of partial sums of a series: if bounded, is absolutely summable by A to S iff, the corresponding series is summed absolutely by G to S. In case of permanent FF and RF-transformation matrix the result analogous to the theorem 2 is well known. Richard has given a converse of this result.

Remarks on theorem 2.2 :

The process given in theorem 2.2 is not reversible. If G is an absolutely permanent RF-transformation matrix and the matrix $P = (p_{nk})$ is defined as $p_{nk} = J_{nk} - J_{nk+1}$, then the transformation by P need not be absolutely permanent. For Example, let $J_{nk} = 1$ for all values of n and k, so that $G = (J_{nk})$ is an absolutely permanent RF-transformation matrix, but when we define $P_{nk} = J_{nk} - J_{nk+1}$ then for all values of n and k, $p_{nk} = 0$ then the matrix $P = (p_{nk})$ so defined doesn't satisfy the condition $\lim_{n\to\infty}\left(\sum_{k=1}^{\infty} p_{nk}\right) = 1$ and according the corresponding transformation is not absolutely permanent.

Theorem 2.3 : Corresponding to every absolutely permanent RF-transformation $G = (g_{nk})$ it is possible to construct absolutely permanent FF transformation matrix $P = (p_{nk})$ by defining $p_{nk} = J_{nk} - g_{n,k+1}$ provided that $\lim_{k\to\infty} g_{nk} = 0$ for each fixed n. We shall need the following basic results in the proof of our theorems.

3. Lemmas :

Lemma 3.1 : The FF -transformation matrix

$$\sigma_n = \sum_{k=1}^{l} p_{nk} s_k \,;\, n = 1, 2, 3, ... \tag{3.1.1}$$

is absolutely permanent iff,

$$\sum_{n=2}^{l} \left| \sum_{j=k}^{l} (p_{nj} - p_{n-1,j}) \right| < M(p) \text{ for } 1, 2, 3, ... \tag{3.1.2}$$

$$\lim_{n\to\infty} p_{nk} = 0 \text{ for } k = 1, 2, ... \tag{3.1.3}$$

$$\lim_{n\to\infty}\left(\sum_{k=1}^{l} p_{nk}\right) = 1 \tag{3.1.4}$$

Lemma 3.2 : The RR- transformation matrix

$$\gamma_n = \sum_{k=1}^{l} b_{nk}u_k\ ,\ n = 1, 2, 3,... \tag{3.2.1}$$

is absolutely permanent iff the conditions

$$\sum_{n=1}^{l} |b_{nk}| < M(B), \tag{3.2.2}$$

$$|b_{nk}| < k_n(B) \tag{3.2.3}$$

Where, the constant $M(B)$, $k_n(B)$ are independent of k, and $\sum_{n=1}^{l} b_{nk} = 1$, for $k = 1, 2,...$ are satisfied.

4. Proof of the theorems are known as follows :

4.1 Proof of the theorem 2.1 :

Let the sequence $\{a_n\}$ be defined as follows -

$$\alpha_1 = \gamma_1,$$

$\alpha_n = \gamma_n - \gamma_{n-1}$ set the matrix $B = [(b)]_{nk}$ for which

$$b_{1k} = g_{1k}\ (k \geq 1)$$

$$b_{nk} = g_{nk} - g_{n-1,k},\ \ (n > 1, k \geq 1) \tag{4.1.1}$$

Then we have,

$$\alpha_n = \sum_{k=1}^{l} b_{nk}u_k,\ n = 1, 2,...$$

The proof of the follows from corresponding conditions of lemma 3.2 for

$$\sum_{n=1}^{l} |b_{nk}| < M(B)$$

Independent of k is the same as

$$\sum_{n=2}^{l} |g_{nk} - g_{n-1,k}| < M(G) \text{ Independent of } k. \tag{4.1.2}$$

Also,

$$\sum_{n=2}^{l} b_{nk} = b_{1k} + \sum_{n=2}^{l} b_{nk}$$

$$= g_{1k} + \sum_{n=2}^{l} (g_{nk} - g_{n-1,k})$$

$$= g_{mk}$$

Hence,

$$\sum_{n=1}^{l} b_{nk} = 1 \text{ For } k = 1, 2, \ldots \text{ is same as}$$

$$\lim_{n \to \infty} g_{mk} = 1 \text{ For } k = 1, 2, \ldots \tag{4.1.3}$$

Finally, by definition (3.1) of (g_{nk}).

We have,

$$g_{nk} = \sum_{j=1}^{n} b_{jk} b_{nk}, \text{ consequently, the condition}$$

$$|b_{nk}| < k_n(B) \text{ is same as}$$

$$|g_{nk}| < k_n(G) \text{ when the constant } k_n \text{ is independent of } k.$$

4.2 Proof of the Theorem 2.2 :

For the proof of the first part of the theorem, we deduce the required condition on the matrix G from the given condition on the matrix P. By hypothesis of the theorem and conditions (3.1.2) of the lemma 3.1,

$$\sum_{n=2}^{l} \left| \sum_{j=k}^{m} (p_{nj} - p_{n-1,j}) \right| = \sum_{n=2}^{l} |g_{nk} - g_{n-1,k}| < M(G) \text{ for } k = 1, 2, \ldots \text{ which } h \text{ is}$$

condition (3.2.1) of theorem 2.1. Also, by hypothesis, we have,

$$g_{nk} = \sum_{j=k}^{l} p_{nj}$$

$$= \sum_{j=1}^{l} p_{nj} - \sum_{j=1}^{k-1} p_{nj}$$

Therefore, by condition (3.1.3) of lemma 3.1,

$$\lim_{n\to\infty} g_{nk} = \lim_{n\to\infty} \left\{ \sum_{j=1}^{l} p_{nj} - \sum_{j=1}^{k-1} p_{nj} \right\} \text{ for } k = 1, 2,...$$

$$= 1$$

i.e. The condition (2.1.3) of Theorem 2.1 is also satisfied from condition (3.1.4) of lemma 3.1, it follows that

$$|g_{nk}| = \left| \sum_{j=k}^{l} p_{nj} \right| < \left| \sum_{j=1}^{l} p_{nj} \right|$$

$$= p_n \text{ say}$$

Hence $|g_{nk}| < k_n(G)$,

where $k_n(G)$ is independent of k. Thus, all three condition of the theorem are reduced. In order to show that the condition of partial sums being bounded is essential in the second part of the theorem.

We consider the following example,

Let, $p_{nk} = 0$ for n and odd k

$= 0$ for even $k < 2n$

$= 2^{n-\frac{k}{2}-1}$ for even k.

This matrix $P = (p_{nk})$ gives absolutely permanent FF-transformation. The element g_{nk} of the corresponding matrix,

$G = (g_{nk})$ as follows -

$$= \sum_{j=k}^{l} p_{nk}$$

$$= \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$$= 1 \text{ for } k$$

$$= 2^{n-\frac{k}{2}-1}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)$$

$$= 2^{n-\frac{k}{2}} \text{ for even } k \text{ and}$$

$$g_{nk} = 2^{n-\frac{k+1}{2}-1}\left(\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots\right)\right)$$

$$= 2^{n-\frac{k+1}{2}} \text{ for odd } k > 2n$$

Thus, for all values of k, when n takes greater than k,

$$g_{nk} = 1$$

i.e. $$\lim_{n\to\infty} g_{nk} = 1 \text{ for } k = 1, 2, 3, \ldots \text{ and,} \tag{4.2.1}$$

$$|g_{nk}| < 2^n < k_n(G) \text{ Independent of } k. \tag{4.2.2}$$

Again,

$$g_{nk} - g_{n-1,k} = 0 \text{ for } k < 2n \Rightarrow 2(n-1),$$

$$= \frac{1}{2} \text{ for } k = 2n,$$

$$= 2^{n-\frac{k+1}{2}-1} \text{ for odd } k > 2n,$$

$$= 2^{n-\frac{k}{2}-1} \text{ for even } k > 2n.$$

Accordingly, we have

$$\sum_{n=2}^{l} |g_{nk} - g_{n-1,k}| = 0 \text{ for } k$$

$$= \sum_{n=1}^{\frac{k-1}{2}} 2^{n-\frac{k+1}{2}-1}$$

$$= 1 - 2^{-\frac{1}{2}(k+1)} < 1 \text{ for odd } k > 2n,$$

$$\sum_{n=2}^{l} |g_{nk} - g_{n-1,k}| = \sum_{k=1}^{\frac{k}{2}} 2^{n-\frac{k}{2}-1}$$

$$= 1 - 2^{-\frac{k}{2}}$$

$$< 1 \text{ for even } k > 2n.$$

Thus, in all cases,

$$\sum_{n=2}^{l} |g_{nk} - g_{n-1,k}| < M(G) \tag{4.2.3}$$

For independent of k.

Hence (4.2.1) - (4.2.3) show that the matrix $G = (g_{nk})$, s completed is the matrix of the theorem 2.1.

The sequence $\{s_n\}$ of the partial sum of the series $\sum_{n=1}^{l} u_n = 2 - 2 + 4 - 4 + 8 - 8 + \cdots$ is not bounded. This sequence $\{s_n\}$ is summed absolutely by p to 0; where as none of the series $\sum_{k=1}^{l} g_{nk} v_k$ converges.

This proves the necessity of the boundness condition.

We have,

$$\sum_{k=1}^{l} p_{nk} s_k = \sum_{k=1}^{m} |g_{nk} - g_{n,k+1}|$$

$$= \sum_{k=1}^{l} g_{nk} u_k - g_{n,m+1} s_m$$

Therefore

$$\sum_{k=1}^{l} p_{nk} s_k = \lim_{m\to\infty} \left\{ \sum_{k=1}^{m} g_{nk} v_k - g_{n,m+1} s_n \right\}$$

$$= \sum_{k=1}^{l} g_{nk} v_k \quad (4.2.4)$$

Since, $g_{n,m+1} \to 0$ as m.

By definition of g_{nk} and $\{s_n\}$ is given to be bounded.

The existence of the expression either side of equality (4.2.4) implies that of the other. Hence, the proof of the theorem is complete.

4.3 Proof of the theorem 2.3 :

The condition (2.1.3) of the theorem 2.1 and the definition of p_{nk} gives

$$\lim_{n\to\infty} p_{nk} = \lim_{n\to\infty} (g_{nk} - g_{n,k+1})$$

$$= 0 \text{ for } k = 1, 2, \ldots$$

This is the condition (3.1.3) of lemma 3.1

Next,

$$\sum_{k=1}^{l} p_{nk} = \sum_{k=1}^{l} ([(g)]_{nk} - g_{n,k+1})$$

$$= g_{n1} + \lim_{k\to\infty} g_{n,k+1}$$

$$= g_{n1}$$

By hypothesis, therefore

$$\lim_{n\to\infty} \left(\sum_{k=1}^{l} p_{nk} \right) = \lim_{n\to\infty} g_{n1} = 1 \text{ by theorem 2.1.}$$

Again,

By definition of p_{nk},

We have,

$$\sum_{n=2}^{l}\left|\sum_{j=k}^{l}(p_{nj}-p_{n-1,j})\right|=\sum_{n=2}^{l}\left|\sum_{j=k}^{l}(\alpha_{n-1,j+1}-\alpha_{n-1,j})\right|$$

Where,

$$\alpha_{nj}=g_{n-1,j}-g_{nj}$$

Now,

$$\lim_{m\to\infty}\left(\sum_{j=k}^{l}(\alpha_{n-1,j+1}-\alpha_{n-1,k})\right)$$

$$=\lim_{m\to\infty}(\alpha_{n-1,m+1}-\alpha_{n-1,k})$$

$$=-\alpha_{n-1,k}$$

Then,

$$\sum_{n=2}^{l}\left|\sum_{j=k}^{l}(p_{nj}-p_{n-1,j})\right|=\sum_{n=2}^{l}|g_{nk}-g_{n-1,k}|<M(G)$$

By condition (2.1.1) of theorem 2.1, the proof of the theorem is thus complete. It may be noted here, that because of the stronger hypothesis QUOTE $\lim_{k\to\infty} g_{nk}=0$ for each fixed n; we did not use the condition (2.1.2) of the theorem 2.1 in the proof of this theorem.

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