The Mathematics Education **ISSN 0047-6269**
Volume - LVI, No. 1, March 2022
Refereed and Peer-Reviewed Quarterly Journal
Journal website : www.internationaljournalsiwan.com

Study and investigations on real life applications of first order differential equation

by **Suresh Kumar Sahani,** *Department of Mathematics, MIT Campus, Janakpurdham - 45600, Nepal*
E-mail : drsks53086@gmail.com
&
Dhirendra Jha, *Department of Mathematics, Janak Hajari Vidhyapeeth, Janakpurdham, Nepal*
E-mail : dhirendrajha349@gmail.com

(Received : February 15, 2022; Accepted : February 25, 2022;
Publication Online : March 23, 2022)

Abstract:

In this research paper, we study and investigate the real life applications of derivatives. This work is motivated by the work of Pielou, E.C. An introduction to mathematical ecology, London-New York-Sydney: Wiley and Sons 1969, Ackerman, E: Biophysical Science. Englewood Cliffs, N.J., Prentice-Hall 1962 and Milhorn, H.T. The application of control theory to physiological systems, Philadelphia-London: Saunders 1966.

Keywords : Cell growth, Differential equation, Radioactive and Tissue.

Introduction :

One of the most simplest differential equation is $\frac{dy}{dt} = ay$ (1)

where a is a constant. The integration is usually performed by a rather symbolic procedure.

Hence, we obtain

$$y = c.\exp(at) \tag{2}$$

We call (2), the general solution of the differential equation.

In the same way, there are so many applications of derivatives in our real life.

Main Results :

1. Growth of a cell : Assume a cell is of mass m_0, in an ideal environment the cell grows. Thus, its mass is a function of time, and we may write $m = m(t)$ with $m = m_0$ at $t = 0$.

Assume that chemicals pass quickly through the cell wall and that growth is only determined by speed of metabolism inside the cell. Since, the output of metabolism depends on the mass of participating molecules. It is reasonable to expect that the growth rate is proportional to the mass at each time instant i.e.,

$$\frac{dm}{dt} \propto m$$

$$\Rightarrow \frac{dm}{dt} = am \tag{3}$$

with certain positive constant a of course, there is limitation: If the mass m of the cell reached a certain size, the cell will divide rather than continue to grow.

Thus, we add a restriction, say $m < m$.

The given differential equation (3) is of the form (1). Thus, the general solution follows from $m = c.\exp(at)$. By our assumption that $m = m_0$ at time instant $t = 0$, we can determine c. We get $c = m_0$.

Hence, its P.I. of (3) is

$$m = m_0 . \exp(at) \tag{4}$$

with above mentioned restriction $m < m$.

With our assumptions we have gone slightly beyond experience. We have introduced some theoretical arguments. It is customary to say that we are model-making. Whether or not our model is biologically meaningful can only be tested by experiments. Here and in subsequent models we share G.F. Gause's view [1] (Gause, 1934, P. 10).

There is no doubt that growth is a biological problem, and that it ought to be solved by experimentation and not at the desk of mathematician. But in order to penetrate deeper into the nature of these phenomena, we must combine the experimental method with mathematical theory, a possibility which has been created by (brilliant researchers). The combination of experimental method with the quantitative theory is in general one of the most powerful tools in the hands of contemporary science. It is worth discussing the above growth model under different aspects.

$\because \frac{dm}{dt}$ was assumed to be proportional to m, we may introduce the relative growth rate defined by $\frac{1}{m} \cdot \frac{dm}{dt}$ (5)

It is quotient of the absolute growth rate $\frac{dm}{dt}$ and the mass m. The differential equation (3) then states: at each time instant, the specific growth rate remains constant.

For this: we assume that a plant which has reached the mass $m = 300$g, grows 12g during the next 24 hours. Then average growth rate is $\frac{12}{24}$ g/hours $= 0.5$g/h. We assume that the growth rate does not fluctuate, we may consider 0.5g/h as a good approximation of the instantaneous growth rate $\frac{dm}{dt}$.

We may ask: Is this growth rate large or small? The answer depends very much on the present mass of the plant. For a plant of mass $m = 10$g only, our growth rate would be tremendous, where as for a large tree of living mass $m = 1000$kg the same growth must be called tiny. Therefore, we have to relate 0.5g/h with the present mass, in our case with 300g.

The quotient is

$$\frac{0.5\text{g/h}}{300\text{g}} = 0.0017\text{h}^{-1}.$$

This quotient is called specific growth rate. With the same specific growth rate, the tree of living mass 1000 kg would gain 1.7 kg/h.

The specific growth rate is an important concept. There are two steps involved. First, when forming $\frac{dm}{dt}$, we relate the increase of mass with time which gives us some measure of velocity of growth. Second, we relate the velocity of growth with the mass present. Let us consider another aspect of the differential equation (4).

With increasing m, the growth rate $\frac{dm}{dt}$ also increases. The growth rate, in turn, determines future values of m, thus, we have a simple example of a feedback mechanism with a single loop :

2. A birth process : Let N stand for the number of individuals an animal or plant population. This number is time dependent so that we may write $N = N(t)$. Strictly speaking, $N(t)$ takes only integral values and is discontinuous function of t. In microorganisms reproduction occurs by simple cell division. In multicellular individuals, we distinguish between vegetative and sexual reproduction. We will include all this possibilities in our study.

We assume that the proportion of reproductive individuals remains constant in growing population. In addition, we assume constant fertility. Then rate of birth is proportional to the number $N(t)$ of individuals.

If we finally exclude death, emigration and immigration, the growth rate coincides with the birth rate.

Thus, $\frac{dN}{dt} = \lambda N$ (6)

Where λ is certain constant.

Referring to the concept introduced in (5), we may λ the specific birth rate.

The differential equation (6) is of type (1).

Hence, $N = N_0 . \exp(\lambda t)$, where N_0 denotes the population size at $t = 0$.

This birth process turns out to be quite realistic in a large population that grows under ideal conditions.

In a small population, we can not expect that the occur renal of birth is distributed evenly overtime. Instead, we face random fluctuations. Then the process has to be modified in the light of probability theory. Such refined model is called a stochastic birth process. See ([2], [3], [4], [5], [6]).

3. Birth-and-death process : Let us consider an animal or plant population under the conditions outlined in the preceding application. Now, we will attend the model by allowing for death. The net change in population size may be positive or negative. Within a time interval of length at, we get

net change = number of births - number of deaths.

$$\therefore \quad \Delta N = \Delta B - \Delta D$$

$$\Rightarrow \quad \frac{\Delta N}{\Delta t} = \frac{\Delta B}{\Delta t} - \frac{\Delta D}{\Delta t}$$

As in previous causes, we treat $N = N(t)$ as a continuous and differentiable function of time even through this means only an approximation to reality. Similarly we assume a large number of births and deaths so that the number of births $B = B(t)$ and of deaths $D = D(t)$ may also be considered as differentiable functions.

Thus, $\frac{dN}{dt} = \frac{dB}{dt} - \frac{dD}{dt}$ (7)

The rate $\frac{dN}{dt}$ may be positive or negative depending on whether occurrence of birth or of death prevails. By hypothesis, the death rate also becomes proportional to $N(t)$.

Thus, $\frac{dB}{dt} = \lambda N$, $\frac{dD}{dt} = \mu N$,

λ denoting the specific birth rate and μ specific death rate.

Hence, $$\frac{dN}{dt} = (\lambda - \mu)N$$

$$\Rightarrow \quad N = N_0 . e^{(\lambda-\mu)t}$$

Where N_0 stands for the population size at time $t = 0$. When the birth rate prevails, the population size increased exponentially. When $\lambda < \mu$, the population size decreases and the population will die out and $\lambda = \mu$ will the population remain stable.

4. Radioactive decay : Let us assume that a substance contains only one sort of radioactive atom. The simplest assumption about decay is that there exist no preferred time for decay and that all atoms have the same chance of disintegration independent of each other. We expect twice as many scintillations per time unit with supply of twice as many atoms, three times as many scintillations with triple amount of atoms, etc. in general, the model requires that the rate of decay is proportional to the number N of radioactive atoms present i.e.

$$\frac{dN}{dt} = -\lambda N \tag{8}$$

where λ is certain positive constant called the decay constant.

Thus $$N = N_0 . e^{-\lambda t}$$

where N_0 devoting the original number of radioactive atoms at time $t = 0$.

5. Living tissue exposed to ionizing radiation : An ionizing beam of particle cosists of either protons, neutrons, deceterons, electrons, γ-ray quanta or the like. If high polymers such as proteins or nucleic acids are hit by an ionizing beam, they may be

irreversibly altered. New bonds may be formed between chains or existing bonds may be broken. We simply say that polymers become damaged.

Let n_0 be the original number of undamaged molecules of a specific chemical compound which are present in cell and which are assumed to be susceptible to radiation. Let D be the number ionizing particles which cross the unit area of the target. We simply call D the dose of radiation. Let n be the n_0 of undamaged molecules after exposure to radiation ($n < n_0$). The question then arises: How does n depend on the dose? When n and D are large numbers, we may operate with these quantities as if they were continuous variables. For this, we assume that n is a function of D. Then we consider the rate $\frac{dn}{dD}$ after exposure to different dose of radiation. Since, a higher dose inflicts more damage, the rate $\frac{dn}{dD}$ must be negative.

When building a model it is plausible to assume that $\frac{dn}{dD}$ is proportional to η.

In this case, we get

$$\frac{dn}{dD} = -s.n \tag{9}$$

where S denotes a certain positive constant. This equation is again of type (i). We notice that the independent variable is not the time but the dose D.

Thus, $n = n_0 e^{-SD}$ (see [7], p. 305)

6. Radioactive tracer : Let us introduce some generalities on a very useful method in biophysics, the compartment analysis. Milhorn ([8], p. 46) defines the compartment in the following way-

If a substance is present in a biological system in several distinguish able forms or locations, and if it passes from one form or location to another form or location at a measurable rate, then each form or location constitutes a separate compartment for the substance. Milhorn illustrates the special case of a single compartment with a tracer dose of radioactive iodine injected into blood stream. Let Q_0 denote the original mass of iodine at time $t = 0$ and denote the mass remaining in the blood at time instant t by $Q = Q(t)$. The blood stream plays the role of the compartment. We assume that iodine is distributed evenly in the entire blood stream

before any loss occurs. Part of the iodine leaves the blood and enters the urine. It is plausible to assume that the rate of loss is proportional to $Q(t)$ at each time instant t. Hence, we may equate this rate to K, Q where K_1 is a certain positive constant. Another part of the iodine enters the thyroid gland at a rate which is also assumed to proportional to $Q(t)$. For this second rate we may write $K_2Q(K_2 > 0)$. The total rate of change is therefore

$$\frac{dQ}{dt} = K_1Q - K_2Q - K_3Q \tag{10}$$

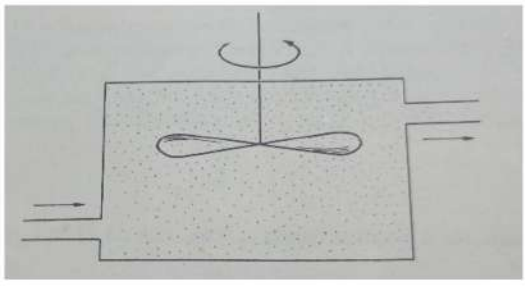
$$\Rightarrow \quad \frac{dQ}{dt} = -KQ \text{ for simplicity,}$$

the solution of the diff. equation is $Q = Q_0e^{-kt}$ (11)

i.e. the concentration of iodine in the blood decreases exponentially.

7. Dilution of a Substance : We consider a second problem that may be approached via compartment analysis.

In a tube containing 2000g of water, 50g of sucrose are dissolved. By stirring, the sucrose will be distributed evenly at all times. Through a pipe, 10g of water flow into tube per minute, and through another pipe, 10g of water leave the tube per minute removing some sucrose at the same time. We may ask: How does the mass of sucrose decrease as a function of time?



Let $M = M(t)$ be the mass of sucrose in the tube. By assumption, we have $M_0 = 50$g at time $t = 0$. In 10g of water the mass of dissolved sucrose is

$$M(t) \cdot \frac{10}{2000} = (0.005)M$$

In a time interval of length Δt, the loss of sucrose from the tube amounts

$$\Delta M = (-0.005)\overline{M}\,\Delta t$$

Where $\overline{M}$ denotes a certain average of $M(t)$ during the time interval. As Δt tends to zero, we get for the rate of decrease

$$\frac{dM}{dt} = (-0.005)M \tag{12}$$

This implies an exponential decrease of sucrose. Our question is answered by the function

$$M = M_0 e^{-(0.005)t} \tag{13}$$

Where t is measured in minutes.

8. Chemical Kinetics : Gaseous nitrogen pentoxide decomposes as stated by the equation

$$2N_2O_5 \rightarrow 4NO_2 + O_2$$

We are interested in the speed of this reaction when the temperature is kept constant. Let $C = [N_2O_5]$ be the concentration of nitrogen pentoxide measure in moles per liter. The concentration $C = C(t)$ is a decreasing function of time so that the derivative dC/dt is negative. This derivative is called the *reaction rate*.

The reaction rate depends on the concentration $C = [N_2O_5]$. Intuitively we expect that the higher the concentration is, the more frequently collisions of two N_2O_5 molecules will occur with the possible emergence of the new bonds NO_2 and O_2. One may theorize that under constant temperature the reaction rate is proportional to C, that is,

$$\frac{dC}{dt} = -kC \tag{14}$$

where k denotes a positive constant. The solution of this differential equation is

$$C = C_0 e^{-kt} \tag{15}$$

C_0 being the concentration of N_2O_5 at time $t = 0$. The experimental facts are in good agreement with this model. As (15) shows the concentration C will asymptotically tend to zero. It will never reach zero exactly.

References :

[1] Gause, G.F. : The struggle for existence. Baltimore: Williams and Wilkins 1934, New York: Hafner, 1964 and 1969.

[2] Baily, N.T.J. : The mathematical theory of epidemics, New-York. London: Hafner. London: Griffin, 1957.

– The elements of stochastic processes with applications to the natural sciences. London-New-York-Sydney: Wiley and Sons, 1964.

– The mathematical approach to biology and medicine. London-New-York-Sydney: Wiley and Sons, 1967.

[3] Chiang, C.L. : Competition and other interactions between species. In: Statistics and mathematics in biology. Kemptorne, O., Bancroft, T.A., Gowen, J.W. Lush, J.L. (Eds.) Chap.14. Reprinted from 1954 edition, New-York-London: Hafner, 1964.

– Introduction to stochastic process in biostatistics. London-New York-Sydney: Wiley and Sons, 1968.

[4] Goel, N.S. and Richter-Dyn, N. : Stochastic models in biology. New York-San Francisco-London: Academic Press, 1974.

[5] Iosifescu, M. and Tautu. P. : Stochastic processes and applications in biology and medicine. Vol. I (theory) Vol. II (models), Biomathematics Vol. III and IV, Berlin-Heidelberg- New York: Springer, 1973.

[6] Pielou, E.C. : An introduction to mathematical ecology, London-New York-Sydney: Wiley and Sons, 1969.

[7] Ackerman, E. : Biophysical Science. Englewood Cliffs, N.J.: Prentice-Hall, 1962.

[8] Milhorn, H.T. : The application of control theory to physiological systems, Philadelphia-London: Saunders, 1966.