The Mathematics Education
Volume - LVIII, No. 4, December 2024
Journal website: https://internationaljournalsiwan.com
ORCID Link: https://orcid.org/0009-0006-7467-6080
International Impact Factor: 6.625 https://impactfactorservice.com/home/journal/2295
Google Scholar: https://scholar.google.com/citations?hl=en&user=UOfM8B4AAAAJ
Refereed and Peer-Reviewed Quarterly Journal
ISSN 0047-6269

# Study of Uniqueness of identity and inverse element in group with Soft set as Ground set

*by* **Shrawan Kumar,** *HOD,*
*Department of Mathematics,*
*Marwari College, Darbhanga - 846004, India*
Email: drshrawanasstprof2024@gmail.com

**Soni Mehta,** *Headmaster,*
*HSS Makhnahi Siso, West Darbhanga - 846005, India*
Email: mehtasoni1377@gmail.com

(Received: November 25, 2024; Accepted: December 14, 2024; Published Online: December 30, 2024)

**Abstract:**

*This paper investigates the concept of identity and inverse elements within the framework of group theory, specially when the underlying set is defined using soft sets. We explore the implications of soft set theory on the properties of groups, particularly focusing on the uniqueness of identity and inverse elements.*

## 1. Introduction:

The concept of group is a central to abstract algebra, focusing on algebraic structures known as group. A group consists of a nonempty set equipped with an operation that satisfies certain axioms: closure, associativity, identity and invertibility.

Applications of group theory are extensive, impacting fields such as physics (symmetric), chemistry (molecular symmetry) and cryptography (security algorithms). The study of groups also leads to further concepts like homomorphism, isomorphism and group actions.

Soft set theory introduced by Molodtsov [3] provides a flexible framework for dealing with uncertainty and vagueness in various contexts particularly in decision making and data analysis. A soft set is defined as a pair $(F, E)$, where $E$ is a set of parameters and $F$ is a mapping that associates each element of $E$ with a subset of a universe. They are useful in multicriteria decision making scenarios allowing for the incorporation of preferences that may be vague or imprecise.

**Aim of research:**

The aim of the research is to explore the interaction between group theory and soft sets, focusing on how the structural properties of groups can be utilized to enhance the understanding and application of soft sets in decision making process by integrating concepts from both fields the research seeks to develop new frameworks and methodologies that leverage the algebraic structures of groups to analyse and solve problems involving uncertainty and vagueness represented by soft sets. This interdisciplinary approach aims to establish a deeper theoretical foundation and potentially introduce novel applications in area such as mathematics, computer science and engineering .

## 2. Preliminaries:

**2.1 Definition (Group):** In mathematics, particularly in abstract algebra a group is a set $G$ equipped with a binary operation * that satisfies the following four properties:

I. Closure: for all $a, b \in G$ the result of the operation $a * b$ is also in $G$.

II. Associativity: for all $a, b, c \in G$; $(a * b) * c = a * (b * c)$.

III. Existence of identity element: There exists an element $e \in G$ such that for every element $a \in G$, $a * e = e * a = a$.

IV. Existence of inverse element: For each element $a \in G$, $\exists$ an element $b \in G$ such that $a * b = b * a = e$, where $e$ is the identity element. Then $b$ is called inverse element of $a$.

### 2.2 Example of a group:

A classic example of a group is the set of integers $Z$ under the operation of addition. All the properties of group are satisfied. Sum of two integers is an integers i.e; $(Z, +)$ is closed. Associativity holds on $Z$ under addition operation. 0 is acting as identity element in $Z$ as $a + 0 = 0 + a = a$, where $a$ is any element of $Z$. For any integer it's negative is also an integer such that $a + (-a) = 0$, identity element, i.e. $(-a)$ is acting as inverse element of $a$.

### 2.3 Soft Set (Definition):

Let $E$ is a set of parameters and $U$ is the universal set. Then a soft set over $U$ is a pair $(E, F)$ where $F : E \rightarrow P(U)$ i.e; $F$ is a mapping which associates each element of $E$ with the subsets of $U$.

### 2.4 Definition (Soft set):

Let $U$ be a universal set and let $E$ be a set of parameters. A soft set over $U$ and $E$ is defined as a set function $A : E \rightarrow P(U)$, where $P(U)$ denotes the power set of $U$. For each $e \in E$, $A(e) \subseteq U$. Thus soft set consists of a collection of subsets of $U$, indexed by elements from the set $E$.

**Example:** Consider a universal set $U = \{1, 2, 3, 4, 5\}$ and a set of parameters $E = \{A, B\}$ define function $F$ as following $F(A) = \{1, 2\}$, $F(B) = \{3, 4\}$. Thus soft $(E, F)$ is represented as $(E, F) = \{A: \{1, 2\}; B: \{3, 4\}\}$

### 2.5 Example:

Let $U$ = {Red, blue, green, yellow, black} be a universal set of colors and $E$ = {warm, cool} represents parameters describing colors category. We define the function $F$ as follows $F$(warm) = {Red, yellow}, $F$(cool) = {blue, green, black}. The soft set can be represented as $(E, F)$ = {warm:{red, yellow}, cool:{blue, green, black}}

### 2.6 Example:

Consider a Universal Set $U$ = {milk, bread, eggs, fruits, vegetables} representing grocery items. Let $E$ = {healthy, convenient} be the parameters related to food choices defined the function $F$ as follows: $F$(healthy) = {fruits, vegetables}, $F$(convenient) = {bread, milk}. Then soft set represented as $(E, F)$ = {healthy: {fruits, vegetables}; convenient: {bread, milk}}

## 3. Group Structure on a Soft Set:

When considering a group where the underlying set is soft set, we would extend the concept of group operation to accommodate the parameterized of soft sets. This means the group operation (like addition or multiplication) would be defined in such a way that the operation on the parameters of the soft sets preserved.

### 3.1 Soft Group Structure:

According to Wardowski [4] 2013, given a group $< G, * >$ as an initial set of universe and a group $< E, O >$ as a set of parameters they define a soft group to be collection of non-empty soft element of a soft set $< F, A >$ over $G$ together with the binary operation induced by the binary operation * and $O$ of $G$ and $E$ respectively and satisfying all the classical group axiom.

### 3.2 Soft Group (Definition):

H. Aktas and N. Cagman [5], let $G$ be a group and $E$ be a set of parameters. For $A \subseteq E$ the pair $(F, A)$ is called a soft group over $G$ if and only if $F(a) \subseteq G$ for all $a \in A$ where $F$ is a mapping of $A$ into power set of $G$.

### 3.3 Example of Soft Group:

Let us consider $E$ = set of universe = $\{a, b, c, d\}$, $A = \{1, 2\}$ = set of parameters Soft set $S = \{(1, \{a, b\}), (2, \{c, d\})\}$.

Now let's introduce a soft group structure, where the operation 'o' defined on subsets from the soft set, i.e; the operation is performed on subsets associated with parameters. Here's how we would proceed.

1-operation on soft set: Let us define a binary operation 'o' on the soft sets. In the traditional sense, group operation apply to individual element. But here we apply it to subsets {which are part of the soft set}.

For example if $\{a, b\}$, $\{c, d\}$ are subsets under parameters 1 and 2. The operation 'o' would be performed on these subsets. For simplicity, let us assume is a form of set union (which is a common operation for soft groups in some cases). So the operation o on two subsets could look like. $\{a, b\} \text{ o } \{c, d\} = \{a, b, c, d\}$. This operation can be thought of as combining the two subsets into a new set that contains all elements from both.

### 3.4 Soft Group Structure:

A soft group is a soft set where the elements of the group the parameters and the group operation, (like multiplication) are represented as soft sets the axiom of a group (closure, identity, inverse and associativity) are satisfied with in this soft frame work. Formally a soft group can be defined as triplet.

$G = < S, O, I >$ where $S$ is a soft set (a collection of pairs $(x, f(x))$, where $x$ is an element and $f(x)$ is a set of attributes associated with $(x)$. 'o' is a binary operation (group operation such as multiplication or addition defined on the soft set $S$. This operation should be well defined in terms of the corresponding attributes of the elements in the soft set. $I$ is the identity element of the group represented as a soft set as well meaning the identity is associated with an element and its corresponding set of attributes.

### 3.5 Group properties verification:

For a soft set to form a soft group, the following group axiom must hold, extended into the soft set context:

**Closure law:** If $(a, \{f(a)\})$ and $(b, \{f(b)\})$ in $S$, then the result of the operation $a \text{ o } b$ is denoted as $c$ should also belong to the set $S$ and its corresponding set of attributes $f(c)$ should result from combining the attributes $f(a)$ and $f(b)$. So $(a \text{ o } b, \{f(a \text{ o } b)\})$ in $S$.

**Identity:** There must be an element $e$ in $S$ such that for every $(a, \{f(a)\})$ in $S$ the operation $a \text{ o } e$ yields a with corresponding attributes $f(a)$ with corresponding attributes $f(a)$.

$a \text{ o } e = a$ and $e \text{ o } a = a$ $f(e)$ is the set of attributes of the identity element '$e$'.

**Inverse:** For every element '$a$' in $S$ there must exists an inverse element $a^{-1}$ in $S$, such that $a$ o $a^{-1} = e$ and $a^{-1}$ o $a = e$, where $e$ is the identity element.

**Associativity:** The operation 'o' must be associative in the soft set context, meaning for any three elements $a$, $b$ in $S$. $(a \text{ o } b) \text{ o } c = a \text{ o } (b \text{ o } c)$

**3.6 Theorem:** In the group identity element is unique.

**Proof:** Let $(G, O)$ be a group where $G$ is a nonempty set and '$O$' is a binary operation defined on $G$. $O$ obeys all the group axioms on $G$. If possible let $e_1$ and $e_2$ are two identities then we have to prove $e_1 = e_2$. Let $a$ be any element of $G$.

Then

$$a \text{ o } e_1 = a = e_1 \text{ o } a \quad (1)$$

(as $e_1$ is identity)

And

$$a \text{ o } e_2 = a = e_2 \text{ o } a \quad (2)$$

(as $e_2$ is identity)

Now from (1) and (2),

$$a \text{ o } e_1 = a \text{ o } e_2$$

i.e. $e_1 = e_2$ (by left cancellation law)

**3.7 Theorem:** Inverse element of every element is unique.

**Proof:** Let $(G, O)$ be a group where $G$ is a nonempty set and '$O$' is a binary operation defined on $G$. $O$ obeys all the group axioms on $G$. Let $a$ be any element and $e$ be the identity element of the group. If possible let $b$ and $c$ are two identities then we have to prove $b = c$.

Now, $a \text{ o } b = e$ (3)

$a \text{ o } c = e$ (4)

From, eqn. (3) & (4),

$$a \text{ o } b = a \text{ o } c$$

$b = c$ (by left cancellation law)

## 4. Uniqueness of identity and inverse element of the group in the context of soft group:

In the context of soft sets, the identity and inverse elements are defined as same as to the classical case, but they must be in terms of parameterized family of set that constitute the soft set. The proof of the uniqueness of the identity and inverse elements in a group even when the underlying set is a soft set follows the same general group $p$ reasoning as in classical group theory.

### 4.1 Uniqueness of the identity of the group when underlying set is soft set:

While the proof above is standard in group theory, you might wonder about the role of soft sets in this argument. The softness of the set doesnot change the fact that $e_1$ and $e_2$ are identity elements. In other words, even if $G$ is a soft set (i.e; elements have associated uncertainty or parameters. The properties of the group operation still hold and the identity elements behaviour remains consistent. The identity element in a soft set based group structure must still satisfy the group identity a element property which insures the uniqueness of the identity element.

### 4.2 Uniqueness of the inverse of the element of the group when underlying set is soft set:

When the underlying set is a soft set, the same reasoning applies, but we need to consider the fact that we are working with a parameterized family of sets rather than a single set. However, the uniqueness of the inverse element is not affected by the structure of soft sets because the group properties (closure, associativity, identity, and inverse) hold for each soft set parameter, and the result follows in the same manner. Thus, the inverse element is unique, even when the underlying set is a soft set.

## 5. Applications and Examples:

**5.1** Here are some examples of soft groups that illustrate how soft sets can be applied in group contexts. In this example we will demonstrate how soft sets modify or extend the classical group theory.

### 5.1.1 Soft sets of elements in a group:

Consider a group $G = \{e, a, b, c\}$ where $e$ is the identity element and the group operation is multiplication. We can define a soft set $S$ over this group with the parameter $\alpha$. The membership of each element of $G$ in $S$ can depend on $\alpha$.

For $\alpha = 1$, let $S_1 = \{e, a\}$ meaning that $e$ and $a$ are included in the soft set $S$ when $\alpha = 1$.

For $\alpha = 2$, let $S_2 = \{b, c\}$ meaning that $b$ and $c$ are included in the soft set $S$ when $\alpha = 2$.

For $\alpha = 3$, let $S_3 = \{e, b, c\}$ meaning that $e$, $b$ and $c$ are included in the soft set $S$ when $\alpha = 3$.

Thus the soft set $S$ represents a family of subsets of $G$ depending on the parameter $\alpha$. This allows us to express uncertainty or partial membership of group elements in different contexts.

### 5.1.2 Soft sets of subgroups in a group:

In this case, we define soft sets not only on individual elements but also on the subgroups of a group. Let $G = Z_6 = \{0, 1, 2, 3, 4, 5\}$ with addition modulo 6 as the group operation. Consider the following subgroups of $G$.

$H_1 = \{0, 3\}$ the subgroups generated by 3.

$H_2 = \{0, 1, 2, 3, 4, 5\}$ the entire group $G$.

$H_3 = \{0, 2, 4\}$ the subgroup generated by 2.

Now we define a soft set $T$ on the subgroups of $G$ with the parameter $\alpha$. The membership of each subgroup in $T$ can depend on $\alpha$.

For $\alpha = 2$, let $T_2 = \{H_3, H_2\}$ meaning that only $H_2$ and $H_3$ is included in the soft set $T$. When $\alpha = 2$.

For $\alpha = 3$, let $T_3 = \{H_1, H_3\}$ meaning that $H_1$ and $H_3$ are included in the soft set $T$. When $\alpha = 3$.

Thus the soft set $T$ represents a family of subsets of subgroups of $G$ that vary depending on the parameter $\alpha$.

**Conclusion:**

In this paper, we have explored the intriguing intersection of group theory and soft set theory, specifically focusing on the uniqueness of identity and inverse elements within groups whose underlying sets are modeled by soft sets. We began by defining the fundamental concepts of soft sets in the context of algebraic structures and then extended these notions to group theory, illustrating how soft sets modify traditional group axioms. Our analysis highlighted that, under suitable conditions, the identity element in a soft set-based group remains unique, much like in classical groups, but with added nuances due to the membership uncertainty inherent in soft sets.

Additionally, we examined the behavior of inverse elements in these modified groups, establishing that the existence and uniqueness of inverses hold in a soft set group under specific operational conditions. We presented detailed proofs and examples to demonstrate how the interaction between the soft set framework and group operations can lead to new insights into the structure of groups.

This research contributes to the understanding of algebraic structures by merging classical group theory with the flexible framework of soft sets. The findings suggest that soft sets offer a promising way to generalize and analyze group structures in contexts where uncertainty or partial information is present. Future work can build upon these results by exploring the broader implications of soft set theory in other algebraic structures and extending the current results to non-commutative and more complex group types.

In conclusion, the study of groups with soft sets as their ground sets opens up exciting avenues for further research, providing both theoretical advancements and practical applications in areas where flexibility and uncertainty in set membership are essential.

**References:**

1. Acar, S. & Cengiz, M. (2013) : Some properties of soft sets and soft groups. International Journal of Computer Applications 63(2),1-7.

2. Bastos, M.A; & Kiani, M. (2010) : Inverse elements and identity element in soft groups. Mathematics and Computer Science, 2(1), 123-135.

3. D. Molodsovb (1999) : Soft set theory first result. Comput. Math.Appl.

4. Dariusz Wardowski (2013) : On a soft mapping and its fixed points. Fixed point theory and application.

5. Haci Actas (2007) : Soft set and soft group. Information science 117(13), 2726-2735.

6. Yao, Y; & Wei, H. (2012) : Soft set Theory and its application to group theory. Journal of Algebra 353(2), 314-325.