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The Steady-State Solution of Serial Queuing Processes with Feedback and Reneging

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Abstract :

O'Brien (1954), Jackson (1954) and Hunt (1955) studied the problems of serial queues in the steady state with Poisson assumptions. In these studies, it is assumed that the unit must go through each service channel without leaving the system. Barrer (1955) obtained the steady-state solution of a single channel queuing model having Poisson input, exponential holding time, random selection where impatient customers leave the service facility after a wait of certain time. Finch (1959) studied simple queues with customers at random for service at a number of service stations in series where the arrival from outside was considered at the initial stage. Feedback is permitted either from the terminal server or from each server of the series to the queue waiting for service at that stage by imposing an upper limit on the number of customers in the system at any time. Singh (1984) studied the problem of serial queues introducing the concept of reneging. Singh and Umed (1994) worked on the network of queuing processes with impatient customers. Punam, Singh and Ashok (2011) found the steady-state solution of serial queuing processes where feedback is not permitted.

In our present work, the steady-state solutions are obtained for serial queuing processes with feedback and reneging in which

(i) The number of serial service channel is M.

(ii) A customer may join any channel from outside and leave the system at any stage after getting service.

(iii) Feedback is permitted from each channel to its previous channel.

(iv) The impatient customer leaves the service facility after wait of certain time.

(v) Poisson arrivals and exponential service times are followed.

(vi) The queue discipline is random selection for service.

(vii) Waiting space is finite with its capacity K.

Keywords : Steady-State, difference-differential, waiting space, random selection, Poisson arrivals, exponential service, feedback and reneging.

1. Formulation of Model :

The system consists of the queues $Q_j(j = 1, 2, 3,...., M)$ with respective servers $S_j(j=1,2,3,....,M)$. Customers demanding different types of service arrive from outside the system with Poisson stream with respective parameters $\lambda_j(j=1, 2,.., M)$ at $Q_j(j=1, 2, 3,...., M)$ respectively. Further the impatient customers after joining any queue may leave the queue without service after a wait of certain time. After the completion of service at S_j, the customer either leave the system with probability p_j or join the next channel with probability q_j or join back the previous channel with probability r_j such that $p_j + q_j + r_j = 1, (j=1, 2, 3,...., M)$. It is being mentioned here that $r_j = 0$ when $j = 1$ as there is no previous channel of the first channel and $q_j = 0$ when $j = M$ since there is no next channel after Mth channel. The service time distribution for severs S_j are mutually independent negative distribution with parameters $\mu_j(j=1, 2, 3,.., M)$.

The applications of such models are of common occurrence. For example, consider the administration of a particular district in a particular state at the level of district head quarter consisting of Block development officer, Tehsildar, Sub-divisional magistrate, District magistrate, etc. Here, the officers of the district correspond to the servers of the model. The people meet the officers of the district in connection with their problems. It is also a common practice that officers call the customers (people) for hearing randomly. The impatient customers after joining the queue may leave the queue without getting service at any stage. The senior officer may send any customer to his junior if some information regarding the customer's problem is lacking.

2. Formulation of Equations :

Define : $P(n_1, n_2, n_3, ..., n_{M-1}, n_M; t)$ = the probability that at time 't' there are n_j customers (which may leave the system after service or join the next phase or join back the previous channel or renege) waiting before $S_j (j = 1, 2, 3, ..., M - 1, M)$.

We define the operators $T_{i\cdot}, T_{\cdot i}, T_{\cdot i, i+1\cdot}, T_{i-1\cdot, \cdot i}$ to act upon the vector $\tilde{n} = (n_1, n_2, n_3, ..., n_M)$ as follows

$$T_{i\cdot}(\tilde{n}) = (n_1, n_2, n_3, ..., n_i - 1, ..., n_M)$$

$$T_{\cdot i}(\tilde{n}) = (n_1, n_2, n_3, ..., n_i + 1, ..., n_M)$$

$$T_{\cdot i, i+1\cdot}(\tilde{n}) = (n_1, n_2, n_3, ..., n_i + 1, n_{i+1} - 1, ..., n_M)$$

$$T_{i-1\cdot, \cdot i}(\tilde{n}) = (n_1, n_2, n_3, ..., n_{i-1} - 1, n_i + 1, ..., n_M)$$

Here we assume that at any instant there are K customers $\left(\sum_{i=1}^{M} n_i = K \right)$ in the system then unit arriving at that instant will not be allowed to join the system and is considered lost for the system.

Following the procedure given by Kelly (1979), we write the difference - differential equations as under

$$\frac{dP(\tilde{n}; t)}{dt} = - \left[\sum_{i=1}^{M} \lambda_i + \sum_{i=1}^{M} \delta(n_i)(\mu_i + C_{in_i}) \right] P(\tilde{n}; t)$$

$$+ \sum_{i=1}^{M} \lambda_i P(T_{i\cdot}(\tilde{n}); t) + \sum_{i=1}^{M} (\mu_i p_i + C_{in_i+1}) P(T_{\cdot i}(\tilde{n}); t)$$

$$+\sum_{i=1}^{M-1}\mu_i q_i P(T_{\cdot i,i+1\cdot}(\tilde{n});t)+\sum_{i=1}^{M}\mu_i r_i P(T_{i-1\cdot,\cdot i}(\tilde{n});t) \tag{2.1}$$

for $n_i \geq 0$ $(i = 1,2,3,....,M)$ and $\sum_{i=1}^{M} n_i < K$.

Where $\delta(x) = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$

$$\frac{dP(\tilde{n};t)}{dt} = -\left[\sum_{i=1}^{M}\delta(n_i)(\mu_i + C_{in_i})\right]P(\tilde{n};t)+\sum_{i=1}^{M}\lambda_i P(T_{\cdot i}(\tilde{n});t)$$

$$+\sum_{i=1}^{M-1}\mu_i q_i P(T_{\cdot i,i+1\cdot}(\tilde{n});t)+\sum_{i=1}^{M}\mu_i r_i P(T_{i-1\cdot,\cdot i}(\tilde{n});t) \tag{2.2}$$

Where $n_i \geq 0$ $(i = 1, 2, 3,...., M)$ and $\sum_{i=1}^{M} n_i = K$.

3. Steady-State Equations :

We write the following Steady-state equations of the queuing model by equating the time derivates to zero in the equations (2.1) and (2.2)

$$\left[\sum_{i=1}^{M}\lambda_i+\sum_{i=1}^{M}\delta(n_i)(\mu_i + C_{in_i})\right]P(\tilde{n}) = \sum_{i=1}^{M}\lambda_i P(T_{i\cdot}(\tilde{n}))+\sum_{i=1}^{M}(\mu_i p_i + C_{in_i+1})\,P(T_{\cdot i}(\tilde{n}))$$

$$+\sum_{i=1}^{M}\mu_i r_i P(T_{i-1\cdot,\cdot i}(\tilde{n}))+\sum_{i=1}^{M-1}\mu_i q_i P(T_{\cdot i,i+1\cdot}(\tilde{n})) \tag{3.1}$$

for $n_i \geq 0$ $(i = 1, 2, 3,...., M)$ and $\sum_{i=1}^{M} n_i < K$

$$\left[\sum_{i=1}^{M}\delta(n_i)(\mu_i + C_{in_i})\right]P(\tilde{n}) = \sum_{i=1}^{M}\lambda_i P(T_{i\cdot}(\tilde{n}))$$

$$+\sum_{i=1}^{M-1}\mu_i q_i P(T_{\cdot i,i+1\cdot}(\tilde{n}))+\sum_{i=1}^{M}\mu_i r_i P(T_{i-1\cdot,\cdot i}(\tilde{n})) \tag{3.2}$$

Where $n_i \geq 0$ $(i = 1, 2, 3, ..., M)$ and $\sum_{i=1}^{M} n_i = K$

4. Steady-State Solutions :

The steady-state solutions of equation (3.1) and (3.2) can be verified as

$$P(\tilde{n}) = P(\tilde{0}) \left(\frac{\left(\lambda_1 + \frac{\mu_2 r_2 \rho_2}{\mu_2 + C_{2n_2+1}}\right)^{n_1}}{\prod_{i=1}^{m_1} (\mu_1 + C_{1i})} \right) \cdot \left(\frac{\left(\lambda_2 + \frac{\mu_1 q_1 \rho_1}{\mu_1 + C_{1n_1+1}} + \frac{\mu_3 r_3 \rho_3}{\mu_3 + C_{3n_3+1}}\right)^{n_2}}{\prod_{i=1}^{m_2} (\mu_2 + C_{2i})} \right) \cdot$$

$$\left(\frac{\left(\lambda_3 + \frac{\mu_2 q_2 \rho_2}{\mu_2 + C_{2n_2+1}} + \frac{\mu_4 r_4 \rho_4}{\mu_4 + C_{4n_4+1}}\right)^{n_3}}{\prod_{i=1}^{n_3} (\mu_3 + C_{3i})} \right) \cdots \left(\frac{\left(\lambda_{M-1} + \frac{\mu_{M-2} q_{M-2} \rho_{M-2}}{\mu_{M-2} + C_{M-2n_{M-2}+1}} + \frac{\mu_M r_M \rho_M}{\mu_M + C_{Mn_M+1}}\right)^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}} (\mu_{M-1} + C_{M-1i})} \right) \cdot$$

$$\left(\frac{\left(\lambda_M + \frac{\mu_{M-1} q_{M-1} \rho_{M-1}}{\mu_{M-1} + C_{M-1n_{M-1}+1}}\right)^{n_M}}{\prod_{i=1}^{n_M} (\mu_M + C_{Mi})} \right) \tag{4.1}$$

With the relation $\sum_{i=1}^{M} n_i \leq K$; $n_i \geq 0$ $(i = 1, 2, 3, ..., M)$

Where

$$\rho_1 = \lambda_1 + \frac{\mu_2 r_2 \rho_2}{\mu_2 + C_{2n_2+1}}, \; \rho_2 = \lambda_2 + \frac{\mu_1 q_1 \rho_1}{\mu_1 + C_{1n_1+1}} + \frac{\mu_3 r_3 \rho_3}{\mu_3 + C_{3n_3+1}},$$

$$\rho_3 = \lambda_3 + \frac{\mu_2 q_2 \rho_2}{\mu_2 + C_{2n_2+1}} + \frac{\mu_4 r_4 \rho_4}{\mu_4 + C_{4n_4+1}} \tag{4.2}$$

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$$\rho_{M\text{-}1} = \lambda_{M\text{-}1} + \frac{\mu_{M\text{-}2}\,q_{M\text{-}2}\,\rho_{M\text{-}2}}{\mu_{M\text{-}2} + C_{M\text{-}2n_{M\text{-}2}+1}} + \frac{\mu_M\, r_M\, \rho_M}{\mu_M + C_{Mn_M+1}}$$

$$\rho_M = \lambda_M + \frac{\mu_{M\text{-}1}\,q_{M\text{-}1}\,\rho_{M\text{-}1}}{\mu_{M\text{-}1} + C_{M\text{-}1n_{M\text{-}1}+1}}$$

Solving these (4.2) M-equations for ρ_M with the help of determinants, we get

$$\rho_M = \frac{\left(\begin{array}{l} \lambda_M\Delta_{M\text{-}1} + \dfrac{q_{M\text{-}1}\mu_{M\text{-}1}\lambda_{M\text{-}1}\Delta_{M\text{-}2}}{(\mu_{M\text{-}1} + C_{M\text{-}1n_{M\text{-}1}+1})} + \dfrac{q_{M\text{-}1}\mu_{M\text{-}1}q_{M\text{-}2}\mu_{M\text{-}2}\lambda_{M\text{-}2}\Delta_{M\text{-}3}}{(\mu_{M\text{-}1} + C_{M\text{-}1n_{M\text{-}1}+1})(\mu_{M\text{-}2} + C_{M\text{-}2n_{M\text{-}2}+1})} + \cdots \\ + \dfrac{q_{M\text{-}1}\mu_{M\text{-}1}}{(\mu_{M\text{-}1} + C_{M\text{-}1n_{M\text{-}1}+1})} \cdot \dfrac{q_{M\text{-}2}\mu_{M\text{-}2}}{(\mu_{M\text{-}2} + C_{M\text{-}2n_{M\text{-}2}+1})} \cdots \dfrac{q_3\mu_3}{(\mu_3 + C_{3n_3+1})}\lambda_3\Delta_2 + \\ \dfrac{q_{M\text{-}1}\mu_{M\text{-}1}}{(\mu_{M\text{-}1} + C_{M\text{-}1n_{M\text{-}1}+1})} \cdot \dfrac{q_{M\text{-}2}\mu_{M\text{-}2}}{(\mu_{M\text{-}2} + C_{M\text{-}2n_{M\text{-}2}+1})} \cdots \dfrac{q_3\mu_3}{(\mu_3 + C_{3n_3+1})}\dfrac{q_2\mu_2}{(\mu_2 + C_{2n_2+1})}\lambda_2\Delta_1 + \\ \dfrac{q_{M\text{-}1}\mu_{M\text{-}1}}{(\mu_{M\text{-}1} + C_{M\text{-}1n_{M\text{-}1}+1})} \cdot \dfrac{q_{M\text{-}2}\mu_{M\text{-}2}}{(\mu_{M\text{-}2} + C_{M\text{-}2n_{M\text{-}2}+1})} \cdots \\ \dfrac{q_3\mu_3}{(\mu_3 + C_{3n_3+1})}\dfrac{q_2\mu_2}{(\mu_2 + C_{2n_2+1})}\dfrac{q_1\mu_1}{(\mu_1 + C_{1n_1+1})}\lambda_1 \end{array}\right)}{\left(\Delta_{M\text{-}1} - \dfrac{q_{M\text{-}1}\mu_{M\text{-}1}}{\mu_{M\text{-}1} + C_{M\text{-}1n_{M\text{-}1}+1}} \cdot \dfrac{r_M\mu_M}{\mu_M + C_{Mn_M+1}}\Delta_{M\text{-}2}\right)} \tag{4.3}$$

Where $\Delta_M = \Delta_{M\text{-}1} - \dfrac{q_{M\text{-}1}\mu_{M\text{-}1}}{\mu_{M\text{-}1} + C_{M\text{-}1n_{M\text{-}1}+1}} \cdot \dfrac{r_M\mu_M}{\mu_M + C_{Mn_M+1}}\Delta_{M\text{-}2}$

$$\Delta_M = \Delta_{M\text{-}2} - \frac{q_{M\text{-}2}\mu_{M\text{-}2}}{\mu_{M\text{-}2} + C_{M\text{-}2n_{M\text{-}2}+1}} \cdot \frac{r_{M\text{-}1}\,\mu_{M\text{-}1}}{\mu_{M\text{-}1} + C_{Mn_{M\text{-}1}+1}}\Delta_{M\text{-}3} \tag{4.4}$$

Continuing in this way

$$\Delta_3 = \Delta_2 - \frac{q_2\mu_2}{\mu_2 + C_{2n_2+1}} \cdot \frac{r_3\mu_3}{\mu_3 + C_{3n_3+1}}$$

Where

$$\Delta_M = \begin{vmatrix} 1 & -\frac{-r_2\mu_2}{\mu_2+C_{2n_2+1}} & 0 & 0 & --- & 0 & 0 & 0 \\ -\frac{q_1\mu_1}{\mu_1+C_{1n_1+1}} & 1 & -\frac{r_3\mu_3}{\mu_3+C_{3n_3+1}} & 0 & --- & 0 & 0 & 0 \\ 0 & -\frac{q_2\mu_2}{\mu_2+C_{2n_2+1}} & 1 & -\frac{r_4\mu_4}{\mu_4+C_{4n_4+1}} & --- & 0 & 0 & 0 \\ - & - & - & - & --- & - & - & - \\ - & - & - & - & --- & - & - & - \\ 0 & 0 & 0 & 0 & --- & -\frac{q_{M\text{-}2}\mu_{M\text{-}2}}{\mu_{M\text{-}2}+C_{M\text{-}2n_{M\text{-}2}+1}} & 1 & -\frac{r_M\mu_M}{\mu_M+C_{Mn_M+1}} \\ 0 & 0 & 0 & 0 & --- & 0 & -\frac{q_{M\text{-}1}\mu_{M\text{-}1}}{\mu_{M\text{-}1}+C_{M\text{-}1n_{M\text{-}1}+1}} & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & -\frac{r_2\mu_2}{\mu_2+C_{2n_2+1}} \\ -\frac{\mu_1 q_1}{\mu_1+C_{1n_1+1}} & 1 \end{vmatrix}$$

$$\Delta_1 = |\,1\,| = 1$$

Since ρ_M is obtained, we can get $\rho_{M\text{-}1}$ by putting the value of ρ_M in the last equation of (4.2), $\rho_{M\text{-}2}$ by putting the values of $\rho_{M\text{-}1}$ and ρ_M in the last but one equation of (4.2). Continuing in this way, we shall obtain $\rho_{M\text{-}3}$, $\rho_{M\text{-}4}$, ..., ρ_3, ρ_2, and ρ_1.

Thus, we write (4.1) as under

$$P(\tilde{n}) = P(\tilde{0})\left(\frac{(\rho_1)^{n_1}}{\prod_{i=1}^{n_1}(\mu_1+C_{1i})}\right)\left(\frac{(\rho_2)^{n_2}}{\prod_{i=1}^{n_2}(\mu_2+C_{2i})}\right)\left(\frac{(\rho_3)^{n_3}}{\prod_{i=1}^{n_3}(\mu_3+C_{3i})}\right)\cdots\cdots\cdots$$

$$\left(\frac{(\rho_{M-1})^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}(\mu_{M-1}+C_{M-1i})}\right)\left(\frac{(\rho_M)^{n_M}}{\prod_{i=1}^{n_M}(\mu_M+C_{Mi})}\right) \tag{4.5}$$

for $n_i \geq 0, (i = 1, 2, 3, \ldots.., M)$

We obtain $P(\tilde{0})$ from the normalizing conditions

$$\sum_{\tilde{n}=\tilde{0}}^{\infty} P(\tilde{n}) = 1 \tag{4.6}$$

and with the restriction that traffic intensity of each service channel of the system is less than unity.

C_{in_i} is the reneging rate at which customer renege after a wait of time T_{0i} whenever there are m_i customer in the service channel Q_i.

$$C_{in_i} = \frac{\mu_{1i} e^{\frac{\mu_{1i}T_{0i}}{n_i}}}{1 - e^{\frac{\mu_{1i}T_{0i}}{n_i}}} \qquad (i = 1,2,3,\ldots.,M)$$

Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the order in which they arrive.

Putting $C_{in_i} = C_i \ (i = 1,2,3,\ldots.,M)$. In the steady-state solution (4.1), the steady-state solution reduces to

$$P(\tilde{n}) = P(\tilde{0})\left(\frac{\rho_1}{\mu_1+C_1}\right)^{n_1}\left(\frac{\rho_2}{\mu_2+C_2}\right)^{n_2}\left(\frac{\rho_3}{\mu_3+C_3}\right)^{n_3}\ldots.$$

$$\ldots\ldots.\left(\frac{\rho_{M-1}}{\mu_{M-1}+C_{M-1}}\right)^{n_{M-1}}\left(\frac{\rho_M}{\mu_M+C_M}\right)^{n_M} \tag{4.7}$$

We obtain $P(\tilde{0})$ from (4.6) and (4.7) as

$$P(\tilde{0}) = \left(1 - \frac{\rho_1}{\mu_1 + C_1}\right)\left(1 - \frac{\rho_2}{\mu_2 + C_2}\right)\left(1 - \frac{\rho_3}{\mu_3 + C_3}\right) \ldots.$$

$$\ldots\ldots \left(1 - \frac{\rho_{M\text{-}1}}{\mu_{M\text{-}1} + C_{M\text{-}1}}\right)\left(1 - \frac{\rho_M}{\mu_M + C_M}\right)$$

Thus $P(\tilde{n})$ is completely determined.

5. Concluding Remarks :

1. The important concept of reneging have been introduced in the present study which is occurring either due to urgent call or due to impatient behaviour of the customer have a bearing effects on the direct as well as indirect cost of the business.

2. The reneging rate depends on time spend to get service, number of customers in a queue and service rate. So, higher the service rate lesser the reneging rate. So, to improve the efficiency at administration department, servers or service rate should be satisfactorily high.

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