Applied Science Periodical **ISSN 0972-5504**
Volume - XXIII, No. 3, August 2021
Refereed and Peer-Reviewed Quarterly Periodical
Journal website : www.internationaljournalsiwan.com

The Study of Serial Channels Connected with Non-Serial Channels with Customer Behaviour in Finite Waiting Space

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Abstract :

O'Brien (1954), Jackson (1954) and Hunt (1955) studied the problems of serial queues in the steady state with Poisson assumptions. In these studies, it is assumed that the unit must go through each service channel without leaving the system. Barrer (1955) obtained the steady-state solution of a single channel queuing model having Poisson input, exponential holding time, random selection where impatient customers leave the service facility after a wait of certain time. Finch (1959) studied simple queues with customers at random for service at a number of service stations in series where the arrival from outside was considered at the initial stage. Feedback is permitted either from the terminal server or from each server of the series to the queue waiting for service at that stage by imposing an upper limit on the number of customers in the system at any time. Singh (1984) studied the problem of serial queues introducing the concept of reneging. Singh and Umed (1994) worked on the network of queuing processes with impatient customers. Punam, Singh and Ashok (2011) found the steady-state solution of serial queuing processes where feedback is not permitted.

In our present work, the steady-state solutions are obtained for serial channel having feedback and balking connected with non-serial queuing channel having reneging and balking in which

(i) M serial channel with feedback and balking are connected with N non-serial channel with reneging and balking.

(ii) A customer may join any channel from outside and leave the system at any stage after getting service.

(iii) Feedback is permitted from each channel to its previous channel in serial channels.

(iv) The customer may balk at each serial and non-serial channel and reneging has been incorporated in non-serial channels only.

(v) The Input process depends upon queue size and Poisson arrivals and exponential service times are followed.

(vi) The queue disciple is random selection for service

(vii) Waiting space is finite.

Keywords : Steady-State, difference-differential, waiting space, serial, non-serial, random selection, Poisson arrivals, exponential service, feedback, balking, reneging, marginal probabilities and mean queue length.

1. Formulation of the Model :

The system consists of the serial queues $Q_j(j=1, 2, 3,...., M)$ and non-serial channels Q_{1i} $(i=1, 2, 3,..., N)$ with respective servers $S_j(j=1, 2, 3,...., M)$ and S_{1i} $(i=1,2,3,...,N)$. Customers demanding different types of service arrive from outside the system in Poisson stream with parameters $\lambda_j(j=1, 2,.., M)$ and $\lambda_{1i}(i=1, 2, 3,..., N)$ at $Q_j(j=1, 2, 3,...., M)$ and $Q_{1i}(i=1, 2, 3,..., N)$ but the sight of long queue at $Q_j(j=1, 2, 3,...., M)$ and $Q_{1i}(i=1, 2, 3,..., N)$ may discourage the fresh customer from joining it and may decide not to enter the service channel at $Q_j(j=1, 2, 3,..., M)$ and $Q_{1i}(i=1, 2, 3,..., N)$. Then the Poisson input rates at $Q_j(j=1, 2, 3,..., M)$ and $Q_{1i}(i=1, 2, 3,..., N)$ would be $\frac{\lambda_j}{n_j+1}$ and $\frac{\lambda_{1i}}{m_i+1}$ where n_j is the queue size of $Q_j(j=1,$

2, 3,..., M) and m_i is the queue size of $Q_{1i}(i=1, 2, 3,..., N)$. Further, the impatient customer joining any service channel Q_{1i} may leave the queue without getting service after wait of certain time. Service time distributions for servers $S_j(j=1, 2, 3,..., M)$ and $S_{1i}(i=1, 2, 3,..., N)$ are mutually independent, negative exponential distribution with parameters $\mu_j(j=1, 2, 3,..., M)$ and $\mu_{1i}(i=1, 2, 3,..., N)$ respectively. After the completion of service at S_j, the customer either leave the system with probability p_j or join the next channel with probability q_j or join back the previous channel with probability r_j such that $p_j+\frac{q_j}{n_{j+1}+1}+\frac{r_j}{n_{j-1}+1}=1$ $(j=1, 2, 3,..., M-1)$ or join any queue $Q_{1i}(i=1, 2, 3,..., N)$ with probability $\frac{q_{Mi}}{m_i+1}$ $(i=1, 2, 3,..., N)$ such that $p_M+r_M+\sum_{i=1}^{N}\frac{q_{Mi}}{m_i+1}=1$. It is being mentioned here that $r_j=0$ for $j=1$ as there is no previous channel of the first channel.

The applications of such models are of common occurrence. For example, consider the administration of a particular district in a particular state at the level of district head quarter consisting of Block Development officer, Tehsildar, Sub-Divisional Magistrate, District Magistrate, etc. These officers correspond to the servers of serial channels. Education Department, Health Department, Irrigation Department, etc. connected with the last server of serial queue corresponds to non-serial channels. The people meet the officers of the district in connection with their problems. It is also a common practice that officers call the customers (people) for hearing randomly. The senior officer may send any customer to his junior if some information regarding the customer's problem is lacking. Further District Magistrate may send the customers to different departments such as Education, Health, Irrigation, etc. if there problems are related to such departments. The customer after seeing long queues before any service channel may decide not to enter the queue. It generally happens that person become impatient after joining the queue and may leave the channel without getting service.

2. Formulation of Equations :

Define : $P(n_1, n_2, n_3,..., n_{M-1}, n_M, m_1, m_2, m_3,..., m_{N-1}, m_N; t)$ = the probability that at time 't' there are n_j customers (which may balk, leave the system after service or join the next phase or join back the previous channel) waiting before S_j ($j = 1, 2, 3, \ldots, M-1, M$) ; m_i customers (which may balk or renege) waiting before the sever S_{1i} ($i = 1, 2, 3, \ldots, N$).

We define the operators $T_{i\cdot}, T_{\cdot i}, T_{\cdot i,i+1\cdot}, T_{i-1\cdot,\cdot i}$ to act upon the vectors $\tilde{n} = (n_1, n_2, n_3, \ldots, n_M)$ or $\tilde{m} = (m_1, m_2, m_3, \ldots, m_N)$ as follows :

$$T_{i\cdot}(\tilde{n}) = (n_1, n_2, n_3, \ldots, n_i - 1, \ldots, n_M)$$

$$T_{\cdot i}(\tilde{n}) = (n_1, n_2, n_3, \ldots, n_i + 1, \ldots, n_M)$$

$$T_{\cdot i, i+1\cdot}(\tilde{n}) = (n_1, n_2, n_3, \ldots, n_i + 1, n_{i+1} - 1, \ldots, n_M)$$

$$T_{i-1\cdot,\cdot i}(\tilde{n}) = (n_1, n_2, n_3, \ldots, n_{i-1} - 1, n_i + 1, \ldots, n_M)$$

Here we assume that at any instant there are K customers in the system i.e.

$$\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j = K$$

Then the customers arriving at that instant will not be allowed to join the system and is considered lost for the system.

Following the procedure given by Kelly (1979), we write the difference-differential equations as

$$\frac{d}{dt} P(\tilde{n}, \tilde{m}; t) = -\left[\sum_{i=1}^{M} \frac{\lambda_i}{n_i + 1} + \sum_{i=1}^{M} \delta(n_i)\mu_i + \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j + 1} + \sum_{j=1}^{N} \delta(m_j)(\mu_{1j} + c_{jm_j}) \right]$$

$$P(\tilde{n}, \tilde{m}; t) + \sum_{i=1}^{M} \frac{\lambda_i}{n_i} P(T_{i\cdot}(\tilde{n}), \tilde{m}; t) + \sum_{i=1}^{M} \mu_i p_i P(T_{\cdot i}(\tilde{n}), \tilde{m}; t)$$

$$+ \sum_{i=1}^{M-1} \frac{\mu_i q_i}{n_{i+1}} P(T_{\cdot i, i+1\cdot}(\tilde{n}), \tilde{m}; t) + \sum_{i=1}^{M} \frac{\mu_i r_i}{n_{i-1}} P(T_{i-1\cdot,\cdot i}(\tilde{n}), \tilde{m}; t)$$

$$+ \sum_{j=1}^{N} \frac{\mu_M q_{Mj}}{m_j} P(n_1, n_2, \ldots, n_M + 1, T_{j\cdot}(\tilde{m}); t)$$

$$+ \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j} P(\tilde{n}, T_{j\cdot}(\tilde{m}); t) + \sum_{j=1}^{N} (\mu_{1j} + c_{jm_j+1}) P(\tilde{n}, T_{\cdot j}(\tilde{m}); t) \quad (2.1)$$

Where $n_i \geq 0\ (i = 1, 2, 3, ..., M),\ m_j \geq 0\ (j = 1, 2, 3, ..., N);\ \delta(x) = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$

and $\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j < K.$

$$\frac{d}{dt} P(\tilde{n}, \tilde{m}; t) = - \left[\sum_{i=1}^{M} \delta(n_i)\mu_i + \sum_{j=1}^{N} \delta(m_j)(\mu_{1j} + c_{jm_j}) \right] P(\tilde{n}, \tilde{m}; t)$$

$$+ \sum_{i=1}^{M} \frac{\lambda_i}{n_i} P(T_{i\cdot}(\tilde{n}), \tilde{m}; t) + \sum_{i=1}^{M-1} \frac{\mu_i q_i}{n_{i+1}} P(T_{\cdot i, i+1\cdot}(\tilde{n}), \tilde{m}; t) + \sum_{i=1}^{M} \frac{\mu_i r_i}{n_{i-1}} P(T_{i-1\cdot, \cdot i}(\tilde{n}), \tilde{m}; t)$$

$$+ \sum_{j=1}^{N} \frac{\mu_M q_{Mj}}{m_j} P(n_1, n_2, ..., n_M + 1, T_{j\cdot}(\tilde{m}); t) + \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j} P(\tilde{n}, T_{j\cdot}(\tilde{m}); t) \qquad (2.2)$$

Where $n_i \geq 0\ (i = 1, 2, 3, ..., M),\ m_j \geq 0\ (j = 1, 2, 3, ..., N)$;

and $\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j = K.$

3. Steady-State Equations :

We write the following Steady-State equations of the queuing model by equating the time derivative to zero in equations (2.1) and (2.2).

$$\left[\sum_{i=1}^{M} \frac{\lambda_i}{n_i + 1} + \sum_{i=1}^{M} \delta(n_i)\,\mu_i + \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j + 1} + \sum_{j=1}^{N} \delta(m_j)(\mu_{1j} + c_{jm_j}) \right] P(\tilde{n}, \tilde{m})$$

$$= \sum_{i=1}^{M} \frac{\lambda_i}{n_i} P(T_{i\cdot}(\tilde{n}), \tilde{m}) + \sum_{i=1}^{M} \mu_i p_i P(T_{\cdot i}(\tilde{n}), \tilde{m})$$

$$+ \sum_{i=1}^{M-1} \frac{\mu_i q_i}{n_{i+1}} P(T_{\cdot i, i+1\cdot}(\tilde{n}), \tilde{m}) + \sum_{i=1}^{M} \frac{\mu_i r_i}{n_{i-1}} P(T_{i-1\cdot, \cdot i}(\tilde{n}), \tilde{m})$$

$$+ \sum_{j=1}^{N} \frac{\mu_M q_{Mj}}{m_j} P(n_1, n_2, ..., n_M + 1,\ T_{j\cdot}(\tilde{m}))$$

$$+ \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j} P(\tilde{n}, T_{j\cdot}(\tilde{m})) + \sum_{j=1}^{N} (\mu_{1j} + c_{jm_j+1})\, P(\tilde{n}, T_{\cdot j}(\tilde{m})) \qquad (3.1)$$

Where $n_i \geq 0\ (i = 1,2,3,....,M),\ m_j \geq 0\ (j = 1,2,3,....,N);\ \delta(x) = \begin{cases} 1 \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$

and $\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j < K.$

$$\left[\sum_{i=1}^{M} \delta(n_i)\mu_i + \sum_{j=1}^{N} \delta(m_j)(\mu_{1j} + c_{jm_j})\right] P(\tilde{n}, \tilde{m})$$

$$= \sum_{i=1}^{M} \frac{\lambda_i}{n_i} P(T_{i\cdot}(\tilde{n}), \tilde{m}) + \sum_{i=1}^{M\text{-}1} \frac{\mu_i q_i}{n_{i+1}} P(T_{\cdot i, i+1\cdot}(\tilde{n}), \tilde{m}) + \sum_{i=1}^{M} \frac{\mu_i r_i}{n_{i\text{-}1}} P(T_{i\text{-}1\cdot, \cdot i}(\tilde{n}), \tilde{m})$$

$$+ \sum_{j=1}^{N} \frac{\mu_M q_{Mj}}{m_j} P(n_1, n_2, ..., n_M + 1, T_{j\cdot}(\tilde{m})) + \sum_{j=1}^{N} \frac{\lambda_{1j}}{m_j} P(\tilde{n}, T_{j\cdot}(\tilde{m})) \qquad (3.2)$$

Where $n_i \geq 0\ (i = 1, 2, 3,...., M),\ m_j \geq 0\ (j = 1,2,3,....,N),$

and $\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j = K.$

4. Steady-State Solutions :

The steady-state solutions of equations (3.1) and (3.2) can be verified as

$$P(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0}) \left(\frac{1}{n_1 !} \left(\frac{\lambda_1 + \frac{\mu_2 r_2 \rho_2}{n_2 + 1}}{\mu_1} \right)^{n_1} \right) \left(\frac{1}{n_2 !} \left(\frac{\lambda_2 + \frac{\mu_1 q_1 \rho_1}{n_1 + 1} + \frac{\mu_3 r_3 \rho_3}{n_3 + 1}}{\mu_2} \right)^{n_2} \right).$$

$$\cdot \left(\frac{1}{n_3 !} \left(\frac{\lambda_3 + \frac{\mu_2 q_2 \rho_2}{n_2 + 1} + \frac{\mu_4 r_4 \rho_4}{n_4 + 1}}{\mu_3} \right)^{n_3} \right) ..... \left(\frac{1}{n_{M\text{-}1} !} \left(\frac{\lambda_{M\text{-}1} + \frac{\mu_{M\text{-}2} q_{M\text{-}2} \rho_{M\text{-}2}}{n_{M\text{-}2} + 1} + \frac{\mu_M r_M \rho_M}{n_M + 1}}{\mu_{M\text{-}1}} \right)^{n_{M\text{-}1}} \right).$$

$$\cdot \left(\frac{1}{n_M !} \left(\frac{\lambda_M + \frac{\mu_{M\text{-}1} q_{M\text{-}1} \rho_{M\text{-}1}}{n_{M\text{-}1} + 1}}{\mu_M} \right)^{n_M} \right) \left(\frac{(\lambda_{11} + \mu_M q_{M1} \rho_M)^{m_1}}{m_1 ! \prod_{j=1}^{m_1} (\mu_{11} + C_{1j})} \right) \left(\frac{(\lambda_{12} + \mu_M q_{M2} \rho_M)^{m_2}}{m_2 ! \prod_{j=1}^{m_2} (\mu_{12} + C_{2j})} \right) ...$$

$$\cdot\left(\frac{(\lambda_{1N}+\mu_M q_{MN}\rho_M)^{m_N}}{m_N!\prod_{j=1}^{m_N}(\mu_{1N}+C_{Nj})}\right) \tag{4.1}$$

With relation $\sum_{i=1}^{M} n_i + \sum_{j=1}^{N} m_j \leq K,\ n_i \geq 0\ (i = 1,2,3,....,M),\ m_j \geq 0\ (j = 1,2,3,....,N)$.

Where

$$\rho_1=\left(\frac{\lambda_1+\frac{\mu_2 r_2\rho_2}{n_2+1}}{\mu_1}\right),\ \rho_2=\left(\frac{\lambda_2+\frac{\mu_1 q_1\rho_1}{n_1+1}+\frac{\mu_3 r_3\rho_3}{n_3+1}}{\mu_2}\right),\ \rho_3=\left(\frac{\lambda_3+\frac{\mu_2 q_2\rho_2}{n_2+1}+\frac{\mu_4 r_4\rho_4}{n_4+1}}{\mu_3}\right)...$$

$$....\ \rho_{M\text{-}1}=\left(\frac{\lambda_{M\text{-}1}+\frac{\mu_{M\text{-}2}q_{M\text{-}2}\rho_{M\text{-}2}}{n_{M\text{-}2}+1}+\frac{\mu_M r_M\rho_M}{n_M+1}}{\mu_{M\text{-}1}}\right),\ \rho_M=\left(\frac{\lambda_M+\frac{\mu_{M\text{-}1}q_{M\text{-}1}\rho_{M\text{-}1}}{n_{M\text{-}1}+1}}{\mu_M}\right) \tag{4.2}$$

Solving these (4.2) M-equations for ρ_M with the help of determinants, we get

$$\rho_M=\left(\frac{\begin{array}{c}\lambda_M\Delta_{M\text{-}1}+\frac{q_{M\text{-}1}\lambda_{M\text{-}1}\Delta_{M\text{-}2}}{n_{M\text{-}1}+1}+\frac{q_{M\text{-}1}}{n_{M\text{-}1}+1}\frac{q_{M\text{-}2}}{n_{M\text{-}2}+1}\lambda_{M\text{-}2}\Delta_{M\text{-}3}+....\\+\frac{q_{M\text{-}1}}{n_{M\text{-}1}+1}\frac{q_{M\text{-}2}}{n_{M\text{-}2}+1}....\frac{q_3}{n_3+1}\lambda_3\Delta_2+\frac{q_{M\text{-}1}}{n_{M\text{-}1}+1}\frac{q_{M\text{-}2}}{n_{M\text{-}2}+1}....\\\frac{q_3}{n_3+1}\frac{q_2}{n_2+1}\lambda_2\Delta_1+\frac{q_{M\text{-}1}}{n_{M\text{-}1}+1}\frac{q_{M\text{-}2}}{n_{M\text{-}2}+1}....\frac{q_3}{n_3+1}\frac{q_2}{n_2+1}\frac{q_1}{n_1+1}\lambda_1\end{array}}{\mu_M\left[\Delta_{M\text{-}1}-\frac{q_{M\text{-}1}}{n_{M\text{-}1}+1}\frac{r_M}{n_M+1}\Delta_{M\text{-}2}\right]}\right) \tag{4.3}$$

where $\Delta_M=\Delta_{M\text{-}1}-\frac{q_{M\text{-}1}}{n_{M\text{-}1}+1}\frac{r_M}{n_M+1}\Delta_{M\text{-}2}$

$$\Delta_{M\text{-}1}=\Delta_{M\text{-}2}-\frac{q_{M\text{-}2}}{n_{M\text{-}2}+1}\frac{r_{M\text{-}1}}{n_{M\text{-}1}+1}\Delta_{M\text{-}3} \tag{4.4}$$

Continuing in this way

$$\Delta_3 = \left(\Delta_2 - \frac{q_2}{n_2+1}\frac{r_3}{n_3+1}\Delta_1\right)$$

Where

$$\Delta_M = \begin{vmatrix} 1 & \frac{-r_2}{n_2+1} & 0 & 0 & - - - - & 0 & 0 & 0 \\ \frac{-q_1}{n_1+1} & 1 & \frac{-r_3}{n_3+1} & 0 & - - - - & 0 & 0 & 0 \\ 0 & \frac{-q_2}{n_2+1} & 1 & \frac{-r_4}{n_4+1} & - - - - & 0 & 0 & 0 \\ - & - & - & - & - - - - & - & - & - \\ - & - & - & - & - - - - & - & - & - \\ 0 & 0 & 0 & 0 & - - - - & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & - - - - & \frac{-q_{M-2}}{n_{M-2}+1} & 1 & \frac{-r_M}{n_M+1} \\ 0 & 0 & 0 & 0 & - - - - & 0 & \frac{-q_{M-1}}{n_{M-1}+1} & 1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & -\frac{r_2}{n_2+1} \\ \frac{-q_1}{n_1+1} & 1 \end{vmatrix} = 1 - \frac{q_1}{n_1+1}\frac{r_2}{n_2+1}$$

$$\Delta_1 = |\,1\,| = 1$$

Since ρ_M is obtained, we can get ρ_{M-1} by putting the value of ρ_M in the last equation of (4.2), ρ_{M-2} by putting the values of ρ_{M-1} and ρ_M in the last but one equation of (4.2) continuing in this way, we shall obtain ρ_{M-3}, ρ_{M-4},...., ρ_3, ρ_2, and ρ_1.

Thus, we write (4.1) as under

$$P(\tilde{n}, \tilde{m}) = P(\tilde{0}, \tilde{0})\left(\frac{1}{n_1!}(\rho_1)^{n_1}\right)\left(\frac{1}{n_2!}(\rho_2)^{n_2}\right)\left(\frac{1}{n_3!}(\rho_3)^{n_3}\right)\cdots\left(\frac{1}{n_{M-1}!}(\rho_{M-1})^{n_{M-1}}\right).$$

$$\cdot\left(\frac{1}{n_M!}(\rho_M)^{n_M}\right)\left(\frac{(\lambda_{11}+\mu_M q_{M1}\rho_M)^{m_1}}{m_1!\prod_{j=1}^{m_1}(\mu_{11}+C_{1j})}\right)\left(\frac{(\lambda_{12}+\mu_M q_{M2}\rho_M)^{m_2}}{m_2!\prod_{j=1}^{m_2}(\mu_{12}+C_{2j})}\right)\left(\frac{(\lambda_{1N}+\mu_M q_{MN}\rho_M)^{m_N}}{m_N!\prod_{j=1}^{m_N}(\mu_{1N}+C_{Nj})}\right)$$

for $n_i \geq 0, (i=1, 2,..., M), m_j \geq 0, (j=1, 2, 3,..., N)$ (4.5)

We obtain $P(\tilde{0}, \tilde{0})$ from the normalizing conditions

$$\sum_{\tilde{n}=\tilde{0},\tilde{m}=\tilde{0}}^{\infty} P(\tilde{n}, \tilde{m}) = 1 \tag{4.6}$$

and with the restriction that traffic intensity of each service channel of the system is less than unity, C_{im_i} is the reneging rate at which customer renege after a wait of time T_{0i} whenever there are m_i customer in the service channel Q_{1i}.

$$C_{im_i} = \frac{\mu_{1i} e^{\frac{\mu_{1i}T_{0i}}{m_i}}}{1 - e^{\frac{\mu_{1i}T_{0i}}{m_i}}} \qquad (i=1,2,3,....N)$$

Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the ordered in which they arrive.

Putting $C_{im_i} = C_i$ $(i=1,2,3,...,N)$. The steady-state solution (4.5) reduces to

$$P(\tilde{n}, \tilde{m}) = P(\tilde{0}, \tilde{0})\left(\frac{1}{n_1!}(\rho_1)^{n_1}\right)\left(\frac{1}{n_2!}(\rho_2)^{n_2}\right)\left(\frac{1}{n_3!}(\rho_3)^{n_3}\right)\cdots\left(\frac{1}{n_{M-1}!}(\rho_{M-1})^{n_{M-1}}\right)$$

$$\left(\frac{1}{n_M!}(\rho_M)^{n_M}\right)\cdot\left(\frac{1}{m_1!}\left(\frac{(\lambda_{11}+\mu_M q_{M1}\rho_M)}{\mu_{11}+C_1}\right)^{m_1}\right)\left(\frac{1}{m_2!}\left(\frac{(\lambda_{12}+\mu_M q_{M2}\rho_M)}{\mu_{12}+C_2}\right)^{m_2}\right)$$

$$\left(\frac{1}{m_3!}\left(\frac{(\lambda_{13}+\mu_M q_{M3}\rho_M)}{\mu_{13}+C_3}\right)^{m_3}\right)\cdots\left(\frac{1}{m_N!}\left(\frac{(\lambda_{1N}+\mu_M q_{MN}\rho_M)}{\mu_{1N}+C_N}\right)^{m_N}\right) \tag{4.7}$$

We obtain $P(\widetilde{0}, \widetilde{0})$ from (4.6) and (4.7) as

$$(P(0,0))^{-1} = \prod_{i=1}^{M} e^{\rho_i} \prod_{j=1}^{N} e^{\rho_{1j}}$$

Where $\rho_{1j} = \dfrac{\lambda_{1j} + \mu_M q_{Mj} \rho_M}{\mu_{1j} + C_j}, \; j = 1, 2, 3, \ldots, N$

Thus $P(\tilde{n}, \widetilde{m})$ is completely determined.

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