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Fixed Points of Kannan and Riech Interpolative Contractions in Double Controlled Metric Spaces

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Abstract:

In this paper, obtained some fixed point theorems in complete double controlled metric spaces. Additionally, These theorems more generalized the results of Deepak et al. [8] and others.

Keywords: *b*-metric space, extended *b*-metric space, controlled metric space, double controlled metric space.

1. Introduction:

Banach contraction principle has long been one of the most important tools in the study nonlinear problems. Banach fixed point theorem has many applications inside and outside Mathematics. In 1989, Bakhtin [1] introduced the concept of *b*-metric space and established many results and many other researchers generalized the Banach fixed point theorem in b-metric space (see [2]-[3]). In 2017, the generalizations of b-metric spaces Kamran et al. [4] and others was introduced extended b-metric spaces by controlling the triangle inequality rather than using control function in the contractive conditions. Mlaiki et al. [7] introduce controlled metric space. Abdeljawad et al. [5] introduced double controlled metric type spaces and prove Banach contraction principle (see [6], [10], [11]). Deepak et al. [8] introduce (λ, a)- interpolative Kannan contraction, introduce (λ, a, b)- interpolative Kannan contraction and introduce (λ, a, b, c)- interpolative Riech contraction and establish some fixed point theorems in complete controlled metric spaces. In this paper, we obtained fixed point theorems in the setting of double controlled metric spaces. These theorems more generalised the results of Deepak et al. [8] and others.

2. Preliminaries:

Definition 2.1 [1]: Let $S \neq \phi$ and $\alpha \geq 1$ be a given real number. Let $L : S \times S \rightarrow [0, +\infty)$ be a function is called b-metric if

1. $L(p, q) \geq 0$,
2. $L(p, q) = 0$ iff $p = q$,
3. $L(p, q) = L(q, p)$,
4. $L(p, q) \leq \alpha [L(p, g) + L(g, q)] \; \forall \, p, q, g \in S$.

A pair (S, L) is called a b-metric space. It is clear that b-metric space is an extension as usual metric space.

Definition 2.2 [4]: Let $S \neq \phi$ and given a function $\alpha : S \times S \rightarrow [1, +\infty)$. Let $L : S \times S \rightarrow [0, +\infty)$ be a function is called extended b-metric if

1. $L(p, q) \geq 0$,
2. $L(p, q) = 0$ iff $p = q$,
3. $L(p, q) = L(q, p)$,
4. $L(p, q) \leq \alpha(p, q)[L(p, g) + L(g, q)] \; \forall \, p, q, g \in S$.

A pair (S, L) is called an extended b-metric space.

Definition 2.3 [7]: Let $S \neq \phi$ and given a function $\alpha : S \times S \rightarrow [1, +\infty)$. Let $L : S \times S \rightarrow [0, +\infty)$ be a function is called controlled metric if

1. $L(p, q) \geq 0$,
2. $L(p, q) = 0$ iff $p = q$,
3. $L(p, q) = L(q, p)$,
4. $L(p, q) \leq \alpha(p, g)[L(p, g) + \alpha(g, q)]\ \forall\, p, q, g \in S$.

A pair (S, L) is called a controlled metric space.

Definition 2.4 [5]: Let $S \neq \phi$ and given a function $\alpha, \beta : S \times S \rightarrow [1, +\infty)$. Let $L : S \times S \rightarrow [0, +\infty)$ be a function is called double controlled metric if

1. $L(p, q) \geq 0$,
2. $L(p, q) = 0$ iff $p = q$,
3. $L(p, q) = L_2(q, p)$,
4. $L(p, q) \leq \alpha(p, g)[L(p, g) + \beta(g, q)L(g, q)]\ \forall\, p, q, g \in S$.

A pair (S, L) is called a double controlled metric space.

A controlled metric type is also a double controlled metric type when taking the same function. The converse is not true in general.

Example 2.1 [5]: Let $S = \{0, 1, 2\}$. Let double controlled type metric $L : S \times S \rightarrow [0, +\infty)$ defined by $L(0, 0) = L(1, 1) = L(2, 2) = 0$, $L(0, 1) = L(1, 0) = 1$, $L(0, 2) = L(2, 0) = ½$, $L(1, 2) = L(2, 1) = 2/5$. And $\theta, \varphi : S \times S \rightarrow [1, +\infty)$ defined by

$\theta(0, 0) = \theta(1, 1) = \theta(2, 2) = \theta(0, 2) = \theta(2, 0) = 1$, $\theta(0, 1) = \theta(1, 0) = 11/10$, $\theta(1, 2) = \theta(2, 1) = 8/5$.

$\varphi(0, 0) = \varphi(1, 1) = \varphi(2, 2) = 1$, $\varphi(0, 2) = \varphi(2, 0) = 3/2$, $\varphi(0, 1) = \varphi(1, 0) = 11/10$, $\varphi(1, 2) = \varphi(2, 1) = 5/4$.

Note that, $L(0, 1) > \theta(0, 2)\, L(0, 2) + \theta(2, 1)\, L(2, 1)$. Thus L is not a controlled metric type for the function θ.

Definition 2.5 [8]: Let (S, L) is called a controlled metric space. Let $H : S \times S$ be self-map. We shall call H $a(\lambda, a)$-interpolative Kannan contraction, if there exist $\lambda \in [0, 1)$, $a \in (0, 1)$ such that

$$L(Hp, Hq) \leq \lambda (L(p, Hp))^a \, (L(q, Hq))^{1-a} \tag{2.1}$$

for all $p, q \in S$ with $p \neq q$.

Definition 2.6 [8]: Let (S, L) is called a controlled metric space. Let $H : S \to S$ be self-map. We shall call H $a(\lambda, a, b)$-interpolative Kannan contraction, if there exist $\lambda \in [0, 1)$, $a, b \in (0, 1)$, $a + b < 1$ such that

$$L(Hp, Hq) \leq \lambda (L(p, Hp))^a \, (L(q, Hq))^b \tag{2.2}$$

for all $p, q \in S$ with $p \neq q$.

Definition 2.7 [8]: Let (S, L) is called a controlled metric space. Let $H : S \to S$ be self-map. We shall call H $a(\lambda, a, b, c)$-interpolative Riech contraction, if there exist $\lambda \in [0, 1)$, $a, b, c \in (0, 1)$, $a + b + c < 1$ such that

$$L(Hp, Hq) \leq \lambda (L(p, q))^a \, (L(p, Hp))^b \, (L(q, Hq))^c \tag{2.3}$$

for all $p, q \in S$ with $p \neq q$.

Definition 2.8 [5]: Let $\{p_n\}$ be a sequence in a double controlled metric space (S, L). Then

1. $\{p_n\}$ is said to be convergent to $p \in S$ written as

$$\lim_{n \to \infty} L(p_n, p) = 0.$$

2. $\{p_n\}$ is said to be Cauchy sequence in S written as

$$\lim_{n, m \to \infty} L(p_n, p_m) = 0.$$

3. (S, L) is said to be complete if every Cauchy sequence is a convergent sequence.

Definition 2.9 [5]: Let (S, L) be a controlled metric space. Let $p \in S$ and $\varepsilon > 0$.

1. The open ball $B(p, \varepsilon)$ is $B(p, \varepsilon) = \{q \in S : L(q, p) < \varepsilon\}$.
2. The mapping $H : S \to S$ is said to be continuous at $p \in S$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $H(B(p, \varepsilon)) \subset B(Hp, \varepsilon)$.

Theorem 2.1 [8]: Let (S, L) be a complete controlled metric space. Let $H : S \to S$ be a (λ, a)-interpolative Kannan contraction. For $p_0 \in S$, take $p_n = T^n p_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \alpha(p_{i+1}, p_{i+2})\, \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+2}) < 1/\lambda.$$

Then T has a unique fixed point.

Theorem 2.2 [8]: Let (S, L) be a complete controlled metric space. Let $H : S \to S$ be a (λ, a, b)-interpolative Kannan contraction. For $p_0 \in S$, take $p_n = T^n p_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \alpha(p_{i+1}, p_{i+2})\, \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+2}) < 1/\lambda.$$

Then T has a unique fixed point.

Theorem 2.3 [8]: Let (S, L) be a complete controlled metric space. Let $H : S \to S$ be a (λ, a, b, c)-interpolative Reich contraction. For $p_0 \in S$, take $p_n = T^n p_0$. Assume that

$$\sup_{m \geq 1} \lim_{i \to \infty} \alpha(p_{i+1}, p_{i+2})\, \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+2}) < 1/\lambda.$$

Then T has a unique fixed point.

3. Main Result:

Our first main result in double controlled metric space as follows.

Theorem 3.1: Let (S, L) be a complete double controlled metric space. Let $H : S \to S$ be a (λ, a)-interpolative Kannan contraction. For $p_0 \in S$, take $p_n = T^n p_0$. Assume that

$$\operatorname*{Sup}_{m\geq 1}\ \lim_{i\to\infty} \beta(p_{i+1}, p_m)\, \alpha(p_{i+1}, p_{i+2})/\alpha(p_i, p_{i+1}) < 1/\lambda \tag{3.1}$$

Then T has a unique fixed point.

Proof: Let $p_0 \in S$ be initial point. Define a sequence $\{p_n\}$ as $p_{n+1} = Hp_n$ for all $n \in N$. Obviously if there exist $n_0 \in N$ for which $p_{n_0+1} = p_{n_0}$, then $Hp_{n_0} = p_{n_0}$ as the proof is finished. Thus we assume $p_{n+1} \neq p_n$

$$L(p_n, p_{n+1}) = L(Hp_{n-1}, Hp_n) \leq \lambda(L(p_{n-1}, Hp_{n-1}))^a\, (L(p_n, Hp_n))^{1-a}$$

$$= \lambda(L(p_{n-1}, p_n))^a\, (L(p_n, p_{n+1}))^{1-a}$$

$$(L(p_n, p_{n+1}))^a \leq \lambda(L(p_{n-1}, p_n))^a \tag{3.2}$$

Since $\alpha < 1$, we have

$$L(p_n, p_{n+1}) \leq \lambda^{1/a}\, L(p_{n-1}, p_n) \leq \lambda L(p_{n-1}, p_n) \leq \lambda^2\, L(p_{n-2}, p_{n-1})$$

$$\leq \lambda^3\, L(p_{n-3}, p_{n-2}) \leq \ldots. \leq \lambda^n\, L(p_0, p_1) \tag{3.3}$$

For all $n, m \in N$ and $n < m$, we have

$$L(p_n, p_m) \leq \alpha(p_n, p_{n+1})\, L(p_n, p_{n+1}) + \beta(p_{n+1}, p_m)\, L(p_{n+1}, p_m)$$

$$\leq \alpha(p_n, p_{n+1})\, L(p_n, p_{n+1}) + \beta(p_{n+1}, p_m)\{\alpha(p_{n+1}, p_{n+2})\, L(p_{n+1}, p_{n+2})$$

$$+ \beta(p_{n+2}, p_m)\, L(p_{n+2}, p_m)\}$$

$$= \alpha(p_n, p_{n+1})\, L(p_n, p_{n+1}) + \beta(p_{n+1}, p_m)\, \alpha(p_{n+1}, p_{n+2})\, L(p_{n+1}, p_{n+2})$$

$$+ \beta(p_{n+1}, p_m)\, \beta(p_{n+2}, p_m)\, L(p_{n+2}, p_m)$$

$$\leq \alpha(p_n, p_{n+1})\, L(p_n, p_{n+1}) + \beta(p_{n+1}, p_m)\, \alpha(p_{n+1}, p_{n+2})\, L(p_{n+1}, p_{n+2})$$

$$+ \beta(p_{n+1}, p_m)\, \beta(p_{n+2}, p_m)\, \alpha(p_{n+2}, p_{n+3})\, L(p_{n+2}, p_{n+3})$$

$$+ \beta(p_{n+1}, p_m)\, \beta(p_{n+2}, p_m)\, \beta(p_{n+3}, p_m)\, L(p_{n+3}, p_m)$$

$$\leq \alpha(p_n, p_{n+1})\, L(p_n, p_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \beta(p_j, p_m) \right) \alpha(p_i, p_{i+1})\, L(p_i, p_{i+1})$$

$$+ \prod_{k=n+1}^{m-1} \beta(p_k, p_m)\, L(p_{m-1}, p_m)$$

$$\leq \alpha(p_n, p_{n+1})\,\lambda^n L(p_0, p_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\beta(p_j, p_m)\right)\alpha(p_i, p_{i+1})\,\lambda^i L(p_0, p_1)$$

$$+ \prod_{k=n+1}^{m-1}\beta(p_k, p_m)\,\lambda^{m-1} L(p_0, p_1)$$

$$\leq \alpha(p_n, p_{n+1})\,\lambda^n L(p_0, p_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\beta(p_j, p_m)\right)\alpha(p_i, p_{i+1})\,\lambda^i L(p_0, p_1) \quad (3.4)$$

Let $S_i = \sum_{i=0}^{i}\left(\prod_{j=0}^{i}\beta(p_j, p_m)\right)\alpha(p_i, p_{i+1})\,\lambda^i L(p_0, p_1)$ (3.5)

Consider $V_i = \left(\prod_{j=0}^{i}\beta(p_j, p_m)\right)\alpha(p_i, p_{i+1})\,\lambda^i L(p_0, p_1)$ (3.6)

We have $V_{i+1}/V_i = \beta(p_{i+1}, p_m)\,\alpha(p_{i+1}, p_{i+2})\lambda/\alpha(p_i, p_{i+1})$

In view of condition (3.1) and the ratio test, we ensure that the series $\sum_i V_i$ converges. Thus $\lim_{n\to\infty} S_n$ exists. Hence, the real sequence $\{S_n\}$ is Cauchy. Now using (3.4), we have

$$L(p_n, p_m) \leq L(p_0, p_1)\,[\lambda^n\,\alpha(p_n, p_{n+1}) + S_{m-1} - S_n] \quad (3.7)$$

Above, we used $\alpha(p, q) \geq 1$. Letting $n, m \to \infty$ in (3.7), we obtain

$$\operatorname{Lim}_{m,n\to\infty} L(p_n, p_m) = 0 \quad (3.8)$$

Thus, the sequence $\{p_n\}$ is a Cauchy in the complete double metric space (S, L), so there is some $p^* \in S$ so the

$$\operatorname{Lim}_{n\to\infty} L(p_n, p^*) = 0 \quad (3.9)$$

That is $p_m \to p^*$ as $n\to\infty$. Now we prove that p^* is a fixed point of H. By (2.1) and condition (4) in def. [2.5], we get

$$L(p^*, Hp^*) \leq \alpha(p^*, p_{n+1})\,L(p^*, p_{n+1}) + \beta(p_{n+1}, Hp^*)\,L(p_{n+1}, Hp^*)$$

$$= \alpha(p^*, p_{n+1})\,L(p^*, p_{n+1}) + \beta(p_{n+1}, Hp^*)\,L(Hp_n, Hp^*)$$

$$\leq \alpha(p^*, p_{n+1})\, L(p^*, p_{n+1}) + \beta(p_{n+1}, Hp^*)\, [\lambda(L(p_n, Hp_n))^a\, (L(p^*, Hp^*))^{1-a}]$$

$$\leq \alpha(p^*, p_{n+1})\, L(p^*, p_{n+1}) + \beta(p_{n+1}, Hp^*)\, [\lambda(L(p_n, p_{n+1}))^a\, (L(p^*, Hp^*))^{1-a}]$$

Taking $n \to \infty$ and using (3.6), (3.7), we obtain that

$$L(p^*, Hp^*) = 0 \tag{3.10}$$

This yields that $p^* = Hp^*$. Now, we prove the uniqueness of p^*. Let q^* be another fixed point of H in S, then $Hq^* = q^*$. Now, by (2.1), we have

$$L(p^*, q^*) = L(Hp^*, Hq^*)$$

$$\leq \lambda(L(p^*, p^*))^a\, (L(q^*, q^*))^{1-a} = 0 \tag{3.11}$$

This yields that $p^* = q^*$. It completes the proof.

Theorem 3.2: Let (S, L) be a complete double controlled metric space. Let $H : S \to S$ be a (λ, a, b)-interpolative Kannan contraction with (3.1) and for $p_0 \in S$, take $p_n = T^n p_0$. Then T has a unique fixed point.

Proof: Following the step of proof of Theorem 3.1, we construct a sequence $\{p_n\}$ as $p_{n+1} = Hp_n$ for all $n \in N$, where $p_0 \in S$ is arbitrary starting point. Then by (2.2), we have

$$L(p_n, p_{n+1}) = L(Hp_{n-1}, Hp_n) \leq \lambda(L(p_{n-1}, Hp_{n-1}))^a\, (L(p_n, Hp_n))^b$$

$$= \lambda(L(p_{n-1}, p_n))^a\, (L(p_n, p_{n+1}))^b$$

$$(L(p_n, p_{n+1}))^{1-b} \leq \lambda(L(p_{n-1}, p_n))^a \tag{3.12}$$

Since $\alpha < 1 - b$, we have

$$L(p_n, p_{n+1})^{1-b} \leq \lambda\, L(p_{n-1}, p_n)^a \leq \lambda\, L(p_{n-1}, p_n)^{1-b}$$

Hence

$$L(p_n, p_{n+1}) \leq \lambda^{1/1-b}\, L(p_{n-1}, p_n) \leq \lambda\, L(p_{n-1}, p_n)$$

and then

$$L(p_n, p_{n+1}) \leq \lambda\, L(p_{n-1}, p_n) \leq \lambda^2\, L(p_{n-2}, p_{n-1})$$
$$\leq \lambda^3 L(p_{n-3}, p_{n-2}) \leq \ldots \leq \lambda^n L(p_0, p_1) \tag{3.13}$$

As already elaborated in the proof of Theorem 3.1, the classical procedure leads to the existence of a fixed point $p^* \in S$. Now, we prove the uniqueness of p^*. Let q^* be another fixed point of H in S, then $Hq^* = q^*$. Now, by (2.2), we have

$$L(p^*, q^*) = L(Hp^*, Hq^*)$$
$$\leq \lambda (L(p^*, p^*))^a\, (L(q^*, q^*))^b = 0 \tag{3.14}$$

This yields that $p^* = q^*$. It completes the proof.

Theorem 3.3: Let (S, L) be a complete double controlled metric space. Let $H : S \to S$ be a (λ, a, b, c)-interpolative Reich contraction with (3.1) and for $p_0 \in S$, take $p_n = T^n p_0$. Then T has a unique fixed point.

Proof: Following the step of proof of Theorem 3.1, we construct a sequence $\{p_n\}$ as $p_{n+1} = Hp_n$ for all $n \in N$, where $p_0 \in S$ is arbitrary starting point. Then by (2.3), we have

$$L(p_n, p_{n+1}) = L(Hp_{n-1}, Hp_n) \leq \lambda (L(p_{n-1}, p_n))^a\, (L(p_{n-1}, Hp_{n-1}))^b\, (L(p_n, Hp_n))^c$$
$$= \lambda (L(p_{n-1}, p_n))^{a+b}\, (L(p_n, p_{n+1}))^c$$

Since $a + b < 1 - c$, the last inequality gives

$$(L(p_n, p_{n+1}))^{1-c} \leq \lambda (L(p_{n-1}, p_n))^{a+b} \leq \lambda (L(p_{n-1}, p_n))^{1-c} \tag{3.15}$$

Hence

$$L(p_n, p_{n+1}) \leq \lambda^{1/1-c}\, L(p_{n-1}, p_n) \leq \lambda\, L(p_{n-1}, p_n)$$

and then

$$L(p_n, p_{n+1}) \leq \lambda L(p_{n-1}, p_n) \leq \lambda^2 L(p_{n-2}, p_{n-1})$$
$$\leq \lambda^3 L(p_{n-3}, p_{n-2}) \leq \ldots \leq \lambda^n L(p_0, p_1) \tag{3.16}$$

As already elaborated in the proof of Theorem 3.1, the classical procedure leads to the existence of a fixed point $p^* \in S$. Now, we prove the uniqueness of p^*. Let q^* be another fixed point of H in S, then $Hq^* = q^*$. Now, by (2.3), we have

$$L(p^*, q^*) = L(Hp^*, Hq^*)$$
$$\leq \lambda(L(p^*, q^*))^a (L(p^*, p^*))^b (L(q^*, q^*))^c = 0 \qquad (3.17)$$

This yields that $p^* = q^*$. It completes the proof.

Remark: Our results extends the corresponding results of Deepak et al. [8].

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