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Fixed Point Theorems of Integral Type Contraction in Cone Controlled Metric Spaces

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Abstract:

In this paper, we present fixed point theorems that generalizes the results of Dinesh et al. [11] in the context of cone controlled metric spaces, specifically focusing on integral type contraction in multivalued mappings.

Keywords: Cone metric space, cone controlled metric space, Integral type contraction, Multivalued mapping

1. Introduction:

In recent years, numerous researchers have explored the strong convergence to a fixed point of contraction mappings with a constant factor in cone metric spaces.

Notably, Seong Hoon Cho and Mi Sun Kim [1] utilized multivalued mappings to establish various fixed-point theorems concerning contractive constants in metric spaces. The concept of cone metric spaces was introduced by Zhang Xian and Huang Guang [2]. Recently, Dinesh et al. [11] presented fixed point theorems for multivalued mappings of integral type contractions within the framework of cone metric spaces.

In this paper, we aim to generalize the results obtained by Dinesh et al. [11] by presenting fixed point theorems in the context of cone controlled metric spaces, specifically focusing on integral type contractions in multivalued mappings.

2. Preliminaries:

Let E be a real Banach space and P be a subset of E. We denote the interior P by int P. The subset P is called a cone if and only if

(i) P is closed, non-empty, and $P \neq \{0\}$.

(ii) $a, b \in R$, $a, b \geq 0$, $p, q \in P$ implies $ap + bq \in P$.

(iii) $P \cap (-P) = \{0\}$.

We define a partial ordering $\leq$ with respect to P by $p \leq q$ if and only if $q - p \in P$. We shall write $p < q$ to indicate that $p \leq q$ but $p \neq q$, write $p < q$ will stand for $q - p \in$ int P. Write $\|.\|$ as a norm on E. The cone P is called normal if there is a number $k > 0$ such that $p, q \in E$, $0 \leq p \leq q$ implies $\|p\| \leq k\|q\|$. The least positive number satisfying the above is called the normal constant P. It is well known that $k \geq 1$. In the following, we always suppose that E is a Banach space, P is a cone in E with int $P \neq 0$ and $\leq$ is a partial ordering with respect to P.

Definition 2.1 [2]: Let $S \neq \varphi$. Suppose that the mappings $d : S \times S \to E$ said to be cone metric if and only if for all $p, q, h \in S$, the following are satisfies:

(i) $0 < d(p, q)$ for all $p, q \in S$ with $p \neq q$ and $d(p, q) = 0$ if and only if $p = q$.

(ii) $d(p, q) = d(q, p)$ for all $p, q \in S$.

(iii) $d(p, q) \leq d(p, h) + d(h, q)$ for all $p, q, h \in S$.

Then pair (S, d) is called a cone metric space.

Example 2.2: Let $E = R^2$, $P = \{(p, q) \in E / p, q \geq 0\} \subset R^2$, $S = R$ and $d : S \times S \to E$ such that

$$d(p, q) = \{|p - q|, \Phi|p - q|\}$$

where $\Phi \geq 0$ is a constant. Then (S, d) is a cone metric space.

Definition 2.3 [3]: Let $S \neq \varphi$ and $S \geq 1$ be a given real number. Suppose that the mappings $d : S \times S \to E$ said to be cone b-metric if and only if for all $p, q, h \in S$, the following are satisfies:

(a) $0 < d(p, q)$ for all $p, q \in S$ with $p \neq q$ and $d(p, q) = 0$ if and only if $p = q$.

(b) $d(p, q) = d(q, p)$ for all $p, q \in S$.

(c) $d(p, q) \leq s[d(p, h) + d(h, q)]$ $p, q, h \in S$.

Then pair (S, d) is called a cone b-metric space.

Example 2.4: Let $E = R^2$, $P = \{(p, q) \in E / p, q \geq 0\} \subset R^2$, $S = R$ and $d : S \times S \to E$ such that

$$d(p, q) = \{|p - q|^x, \phi|p - q|^x\}$$

where $\phi \geq 0$ and $x > 1$ are two a constant.

Then (S, d) is a cone metric space, but not a cone metric space. In fact, we only need to prove (c) in definition 2.4 as follows: Let $p, q, r \in S$. Set $u = p - h$, $v = h - q$, so $p - q = u + v$. From inequality $(a + b)^x \leq (2\max\{p, q\})^x \leq 2^x(a^x + b^x)$ for all $a, b \geq 0$,

we have $$|p - q|^x = |u + v|^x \leq 2^x(|u|^x + |v|^x) = 2^x(|p - h|^x + |h - q|^x),$$

which implies that,

$$d(p, q) \leq s[d(p, h) + d(h, q)] \text{ with } s = 2^x > 1.$$

But $|p - q|^x \leq |p - h|^x + |h - q|^x$ is impossible for all $p > h > q$. Taking the account of the inequality $(a + b)^x > a^x + b^x$ all $a, b > 0$, we arrive at

$$|p - q|^x = |u + v|^x = (u + v)^x > u^x + v^x = (p - h)^x + (h - q)^x = |p - h|^x + |h - q|^x,$$

for all $p > h > q$. Thus Definition 2.1 is not satisfied, hence (S, d) is not a cone metric space.

Definition 2.5 [3]: Let $S \neq \varphi$. Suppose that the mappings $d : S \times S \to E$ and $\alpha : Z \times Z \to [1, \infty)$ be control functions said to be cone controlled metric if and only if for all $p, q, h \in S$, the following are satisfies:

(a) $0 < d(p, q)$ for all $p, q \in S$ with $p \neq q$ and $d(p, q) = 0$ if and only if $p = q$.

(b) $d(p, q) = d(q, p)$ for all $p, q \in S$.

(c) $d(p, q) \leq [\alpha(p, h)\, d(p, h) + \alpha(h, q)\, d(h, q)]$; $p, q, h \in S$.

Then pair (S, d) is called a cone controlled metric space.

Example 2.6: Let $S = \{1, 2, 3\}$ as $\alpha : S \times S \to [1, \infty)$, $\alpha(p, q) = 1 + p + q$. Consider the real Banach algebra $E = R^2$ with solid cone $k = \{(p, q) \in R^2 : p, q \geq 0\}$. Assume that we define $d : S \times S \to E$ by

$$d(1, 2) = d(2, 1) = (100, 100),\ d(1, 3) = d(3, 1) = (1200, 1200),$$

$$d(3, 2) = d(2, 3) = (800, 800),\ d(1, 1) = d(2, 2) = d(3, 3) = (0, 0) = 0.$$

Clearly the first and second condition of a controlled cone metric type space over a Banach algebra E are satisfied. For the third condition, we have

$$\alpha(1, 3)\, d(1, 3) + \alpha(3, 2)\, d(3, 2) - d(1, 2) = 5(1200, 1200) + 6(800, 800) - (100, 100) = (10700, 10700)$$

$$\alpha(1, 2)\, d(1, 2) + \alpha(2, 3)\, d(2, 3) - d(1, 3) = 4(100, 100) + 6(800, 800) - (1200, 1200) = (4000, 4000)$$

$$\alpha(2, 1)\, d(2, 1) + \alpha(1, 3)\, d(1, 3) - d(2, 3) = 4(100, 100) + 5(1200, 1200) - (600, 600) = (5800, 5800)$$

hence for all $p, q, h \in E$ by $d(p, q) \leq \alpha(p, h)\, d(p, h) + \alpha(h, q)\, d(h, q)$.

Thus (S, d) is a controlled cone metric space over a Banach algebra E.

Definition 2.7 [3]: Let (S, d) be a cone controlled metric space, $p \in S$ and $\{p_n\}$ be a sequence in S. Then

(i) $\{p_n\}$ converges to p whenever, for every $\lambda \in E$ with $0 << \lambda$, there is a natural number N show that $d(p_n, p) << \lambda$ for all $n \geq N$.

(ii) $\{p_n\}$ is a Cauchy sequence, whenever, for every $\lambda \in E$ with $0 << \lambda$, there is a natural number N such that $d(p_n, p_m) << \lambda$ for all $m, n \geq N$.

(iii) $\{p_n\}$ is a complete cone metric space if every Cauchy sequence is convergent.

Definition 2.8: Let (S, d) be a metric space. If every Cauchy sequence is convergent in S, then (S, d) be a complete cone metric space. The family of non-empty closed bounded subset of X is denoted as CB(S). The Hausdorff distance on CB(S) is denoted by $H(.,.)$. Implied

$$A, B \in \mathrm{CB}(S)$$

$$H(A, B) = \max \{\sup_{\theta_1 \varepsilon A} d(\theta_1, B), \sup_{\theta_2 \varepsilon B} d(A, \theta_2)\}$$

The distance between the point θ_1 and the subset b is given by $(\theta_1, B) = \inf \{d(\theta_1, \theta_2); \theta_2 \in B\}$.

If $p_1 \in T(S)$, then an element $p \in S$ is considered a fixed point in the multi-valued mapping $T : S \to 2^S$.

Definition 2.9: If the sets $\{p_{t-1}, p_t\}_{t=1}^{n}$ are pairwise disjoint and $[\theta_1, \theta_2] = U_{t=1}^{n} [p_{t-1}, p_t) \, u \{\theta_2\}$.

Then the set $\{\theta_1 = p_0 \, p_1 \, p_3 \, ... \, p_n = \theta_2\}$ is referred to as a partition for $[\theta_1, \theta_2]$.

Definition 2.10: Cone lower summation and cone upper summation are defined as follows for each partition Q of $[\theta_1, \theta_2]$ and each rising function $g : [\theta_1, \theta_2] \to P$

$$L_n^{\mathrm{con}}(g, Q) = \sum_{t=0}^{n-1} g(p_t) \, \| p_t - p_{t+1} \|$$

$$U_n^{\mathrm{con}}(g, Q) = \sum_{t=0}^{n-1} g(p_{t+1}) \, \| p_t - p_{t+1} \| \text{ correspondingly.}$$

Definition 2.11: Assume that P in E is a normal cone $g : [\theta_1, \theta_2] \to P$ is called with respect to cone P or simplicity on an integrable function on $g : [\theta_1, \theta_2]$. Cone integrable function iff

$$\operatorname*{Lim}_{n\to\infty} L_n^{\text{con}}(g, Q) = S^{\text{con}} = \operatorname*{Lim}_{n\to\infty} U_n^{\text{con}}(g, Q)$$

where S^{con} necessary be unique, for any partition Q of θ_1, θ_2. By using $\int_{\theta_2}^{\theta_1} g(p)\, d_P(p)$, we demonstrate the common value S^{con}, simplifying it to $\int_{\theta_2}^{\theta_1} gd_P$.

Definition 2.12: If and only if, for any $\theta_1, \theta_2 \in P$, the function $g : P \to E$ is refered to be a Sub additive cone integrable function.

$$\int_0^{\theta_1+\theta_2} gd_P \le \int_0^{\theta_1} gd_P + \int_0^{\theta_2} gd_P$$

Definition 2.13: Let $E = S = R$, $d(p, q) = |p - q|$, $P = (0, \infty)$ and $g(t) = 1/t + 1$ for all $t > 0$. Then for all $\theta_1, \theta_2 \in P$,

$$\int_0^{\theta_1+\theta_2} \frac{dt}{(t+1)} = \text{In}(\theta_1 + \theta_2 + 1), \int_0^{\theta_1} \frac{dt}{(t+1)} = \text{In}(\theta_1 + 1), \int_0^{\theta_2} \frac{dt}{(t+1)} = \text{In}(\theta_2 + 1)$$

Since $\theta_1, \theta_2 \ge 0$, then $\theta_1 + \theta_2 + 1 \le \theta_1 + \theta_2 + 1 + \theta_1\theta_2 = (\theta_1 + 1)(\theta_2 + 1)$. Therefore

$$\text{In}(\theta_1 + \theta_2 + 1) \le \text{In}(\theta_1 + 1) \le \text{In}(\theta_2 + 1)$$

This demonstrates that g is an illustration of a subadditive cone integrable function.

Lemma 2.1: Let P be a cone and $\{p_n\}$ be a sequence in E, if $\lambda \in \text{int}\, P$ and $0 \le p_n \to 0$ as $n \to \infty$, then there exists N such that all $n > N$, we have $p_n << \lambda$.

Lemma 2.2: Let $p, q, h \in E$, if $p \le q$ and $q << h$, then $p << h$.

Lemma 2.3: Let P be a cone and $0 \le p << \lambda$ for each $\lambda \in \text{int}\, P$ then $p = 0$.

Lemma 2.4: Let P be a cone, if $p \in P$ and $p \le kp$ for some $0 \le k < 1$, then $p = 0$.

Lemma 2.5: Let P be a cone and $a \le b + c$ for each $\lambda \in \text{int}\, P$.

3. Main Result:

Theorem 3.1: Let $(S, \pounds)$ be a complete cone controlled metric space and a multi-valued mapping $T : S \to CB(S)$ and $\alpha : S \times S \to [1, \infty)$ that is satisfied for every p, $q \in S$. For any $\varepsilon > 0$, $\int_0^{\varepsilon} g(t)dt >> 0$ in the definition of the function $g : P\cup\{0\} \to P\cup\{0\}$.

$\int_0^{H(Tp,Tq)} g(t)dt \leq \theta_1 \int_0^{d(p,Tp)+d(q,Tq)} g(t)dt + \theta_2 \int_0^{d(p,Tq)+d(q,Tp)} g(t)dt$ for all $p, q \in S$

we have, $2\theta_1 + \theta_2\alpha(p_{n-1}, p_n) + \theta_2\alpha(p_n, p_{n+1}) < 1$, $\lim_{n\to\infty} \alpha(p, p_n)$, $\lim_{n\to\infty} \alpha(p_n, p)$ and α satisfies the following conditions

$$\operatorname{Sup}_{m\geq 1} \lim_{i\to\infty} \alpha(p_{i+1}, p_{i+2})\ \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+1}) < 1/k$$

where $k \in (0,1)$. Then T has a unique fixed point in S.

Proof: For all $p_0 \in S, n \geq 1, p_1 \in Tp_0$ and $p_{n+1} \in Tp_n$

$$\int_0^{d(p_{n+1},p_n)} g(t)dt \leq \int_0^{H(Tp_n, Tp_{n-1})} g(t)dt \leq \theta_1 \int_0^{d(p_n, Tp_n)+d(p_{n-1}, Tp_{n-1})} g(t)dt$$
$$+ \theta_2 \int_0^{d(p_n, Tp_{n-1})+d(p_{n-1}, Tp_n)} g(t)dt$$
$$\leq \theta_1 \int_0^{d(p_n,p_{n+1})+d(p_{n-1},p_n)} g(t)dt + \theta_2 \int_0^{d(p_n,p_n)+d(p_{n-1},p_{n+1})} g(t)dt$$
$$\leq \theta_1 \int_0^{d(p_n,p_{n+1})+d(p_{n-1},p_n)} g(t)dt + \theta_2 \int_0^{d(p_{n-1},p_{n+1})} g(t)dt$$
$$\leq \theta_1 \int_0^{d(p_n,p_{n+1})+d(p_{n-1},p_n)} g(t)dt + \theta_2 \int_0^{\alpha(p_{n-1},p_n)d(p_{n-1},p_n)+\alpha(p_n,p_{n+1})d(p_n,p_{n+1})} g(t)dt$$
$$(1 - \theta_1 - \theta_2(p_n, p_{n+1})) \int_0^{d(p_n,p_{n+1})} g(t)dt \leq (\theta_1 + \theta_2(p_{n-1}, p_n)) \int_0^{d(p_{n-1},p_n)} g(t)dt$$
$$\int_0^{d(p_n,p_{n+1})} g(t)dt \leq [(\theta_1 + \theta_2(p_{n-1}, p_n))/(1 - \theta_1 - \theta_2(p_n, p_{n+1}))] \int_0^{d(p_{n-1},p_n)} g(t)dt\ \theta_2.$$

Let $[(\theta_1 + \theta_2(p_{n-1}, p_n))/(1 - \theta_1 - \theta_2(p_n, p_{n+1}))] = k < 1$

$$\int_0^{d(p_n,p_{n+1})} g(t)dt \leq k \int_0^{d(p_{n-1},p_n)} g(t)dt\ \theta_2.$$
$$\leq k^2 \int_0^{d(p_{n-2},p_{n-1})} g(t)dt\ \theta_2.$$
$$\leq \ldots$$
$$\leq k^n \int_0^{d(p_0,p_1)} g(t)dt\ \theta_2.$$

For $n > m$ we have

$$\int_0^{d(p_n,p_m)} g(t)dt \leq \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)d(p_{n+1},p_m)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)\alpha(p_{n+1},p_{n+2})d(p_{n+1},p_{n+2}) +\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m)d(p_{n+1},p_m)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)\alpha(p_{n+1},p_{n+2})d(p_{n+1},p_{n+2})+\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m) \alpha(p_{n+2},p_{n+3})d(p_{n+1},p_{n+3})+\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m)\alpha(p_{n+3},p_m)d(p_{n+3},p_m)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(p_j,p_m)\right)\alpha(p_i,p_{i+1})d(p_i,p_{i+1}) + \prod_{i=n+1}^{m-1}\alpha(p_i,p_m)\alpha(p_{m-1},p_m)d(p_{m-1},p_m)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})k^n d(p_0,p_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(p_j,p_m)\right)\alpha(p_i,p_{i+1})k^i d(p_0,p_1) + \prod_{i=n+1}^{m-1}\alpha(p_i,p_m)\alpha(p_{m-1},p_m)k^{m-1} d(p_0,p_1)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})k^n d(p_0,p_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\alpha(p_j,p_m)\right)\alpha(p_i,p_{i+1})k^i d(p_0,p_1)} g(t)dt$$

Let $S_i = \sum_{i=0}^{i}\left(\prod_{j=0}^{i}\alpha(p_j, p_m)\right)\alpha(p_i, p_{i+1})k^i\, d(p_0, p_1)$

$$\int_0^{d(p_n,p_m)} g(t)dt \leq \int_0^{d(p_0,p_1)\,[k^n\,\alpha(p_n,p_{n-1})+s_{m-1}-s_n]} g(t)dt$$

$$\leq [k^n\,\alpha(p_n, p_{n-1}) + s_{m-1} - s_n]\int_0^{d(p_0,p_1)} g(t)dt$$

Now, using the feel the $\alpha(p, q) \geq 1$ and by using the ratio test $\lim_{n\to\infty} S_n$ exists.

Hence $\{S_n\}$ is a Cauchy sequence, eventually $n, m \to \infty$ then $\int_0^{d(p_n,p_m)} g(t)dt << \lambda$.

Hence $\{p_n\}$ is a Cauchy sequence.

Let 0 is an interior of λ be given, there exists $N_1 \in N$ such that

$[k^n\,\alpha(p_n, p_{n-1}) + s_{m-1} - s_n]\int_0^{d(p_0,p_1)} g(t)dt << \lambda$ for all $m \geq N_1$ this gives

$\int_0^{d(p_0,p_1)} g(t)dt << \lambda$.

For $n \geq m$, $\{p_n\}$ is a Cauchy sequence in $(S, £)$ is complete cone controlled metric space, there exists $y \in S$ such that $p_n \to y$.

There exists $N_2 \in N$ such that $\int_0^{d(p_n,y)} g(t)dt << \lambda/2$, for all $n \geq N_2$.

Hence for $n \geq N_2$ we have $\int_0^{d(p_n,y)} g(t)dt << \lambda/2$,

$$\int_0^{d(Ty,y)} g(t)dt \leq \alpha(Tp_n, Ty) \int_0^{Hd(Tp_n, Ty)} g(t)dt + \alpha(Tp_n, y) \int_0^{d(Tp_n,y)} g(t)dt$$

$$\leq \alpha(p_{n+1}, Ty)\, \theta_1 \int_0^{d(p_n,p_{n+1})+d(y,Ty)} g(t)dt + \alpha(p_{n+1}, Ty)$$

$$\theta_2 \int_0^{d(p_n,y)+d(y,Tp_n)} g(t)dt + \alpha(p_{n+1}, y) \int_0^{d(p_{n+1},y)} g(t)dt$$

$$\int_0^{d(Ty,y)} g(t)dt \leq \frac{\alpha(p_{n+1},Ty)\, \theta_1 \alpha(p_n, y) + \alpha(p_{n+1},Ty)\, \theta_2 \alpha(p_n, y)}{1 - \alpha(p_{n+1},Ty)\, \theta_1 - \alpha(p_{n+1},Ty)\, \theta_2 \alpha(y,Ty)} \int_0^{d(p_n,y)} g(t)dt$$

$$+ \frac{\alpha(p_{n+1},Ty)\, \theta_1 \alpha(p_{n+1}, y) + \alpha(p_{n+1},Ty)\, \theta_2 + \alpha(p_n, y)}{1 - \alpha(p_{n+1},Ty)\, \theta_1 - \alpha(p_{n+1},Ty)\, \theta_2 \alpha(y,Ty)} \int_0^{d(p_{n+1},y)} g(t)dt << \lambda$$

For all $n \geq N_2$, $\int_0^{d(y,Ty)} g(t)dt << \lambda/r$, for $r \geq 1$ we get $(\lambda/r) - 1 \int_0^{d(y,Ty)} g(t)dt \in P$ and $r \to 0$, we get $\lambda/r \to 0$ and P is closed $\int_0^{-d(y,Ty)} g(t)dt \in P$ but $\int_0^{d(y,Ty)} g(t)dt \in P$. Implies, $\int_0^{d(y,Ty)} g(t)dt = 0$ and so $y \in Ty$.

Corollary 3.2: Let $(S, £)$ be a complete cone controlled metric space and a multi-valued mapping $T : S \to CB(S)$ and $\alpha : S \times S \to [1, \infty)$ that is satisfied for every p, $q \in S$. For any $\varepsilon > 0$, $\int_0^{\varepsilon} g(t)dt >> 0$ in the definition of the function $g : P \cup \{0\} \to P \cup \{0\}$.

$$\int_0^{d(Tp,Tq)} g(t)dt \leq \theta_1 \int_0^{d(p,Tp)+d(q,Tq)} g(t)dt \text{ for all } p, q \in S \text{ and } \theta_1 \in [0, 1).$$

$\lim_{n\to\infty} \alpha(p, p_n)$, $\lim_{n\to\infty} \alpha(p_n, p)$ and α satisfies the following conditions

$$\text{Sup}_{m\geq 1} \lim_{i\to\infty} \alpha(p_{i+1}, p_{i+2})\, \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+1}) < 1/k$$

where $k \in (0, 1)$. Then T has a unique fixed point in S.

Proof: The corollary proof can be obtained by simply setting $\theta_2 = 0$ in the preceding theorem.

Theorem 3.3: Let $(S, \pounds)$ be a complete cone controlled metric space and a multi-valued mapping $T : S \to CB(S)$ and $\alpha : S \times S \to [1, \infty)$ that is satisfied for every $p, q \in S$. For any $\varepsilon > 0$, $\int_0^{\varepsilon} g(t)dt >> 0$ in the definition of the function $g : P\upsilon\{0\} \to P\upsilon\{0\}$.

$$\int_0^{H(Tp,Tq)} g(t)dt \le \mu \int_0^{\max\{d(p,q)d(p,Tp),d(q,Tq)\}} g(t)dt \text{ for all } p, q \in S \text{ and } \mu \in [0,1).$$

$\lim_{n\to\infty} \alpha(p, p_n)$, $\lim_{n\to\infty} \alpha(p_n, p)$ and α satisfies the following conditions

$$\mathrm{Sup}_{m\ge 1} \lim_{i\to\infty} \alpha(p_{i+1}, p_{i+2})\, \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+1}) < 1/k$$

where $k \in (0, 1)$. Then T has a unique fixed point in S.

Proof: For all $p_0 \in S$, $n \ge 1$, $p_1 \in Tp_0$ and $p_{n+1} \in Tp_n$

$$\int_0^{d(p_{n+1},p_n)} g(t)dt \le \int_0^{H(Tp_n,Tp_{n-1})} g(t)dt \le \mu \int_0^{\max\{d(p_n,p_{n-1})d(p_n,Tp_n),d(p_{n-1},Tp_{n-1})\}} g(t)dt$$

$$\le \mu \int_0^{\max\{d(p_n,p_{n-1})d(p_n,p_{n+1}),d(p_{n-1},p_n)\}} g(t)dt$$

$$\le \mu \int_0^{d(p_n,p_{n-1})} g(t)dt \le \mu^n \int_0^{d(p_1,p_0)} g(t)dt$$

For $n > m$ we have

$$\int_0^{d(p_n,p_m)} g(t)dt \le \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)d(p_{n+1},p_m)} g(t)dt$$

$$\le \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)\alpha(p_{n+1},p_{n+2})d(p_{n+1},p_{n+2}) +\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m)d(p_{n+1},p_m)} g(t)dt$$

$$\le \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)\alpha(p_{n+1},p_{n+2})d(p_{n+1},p_{n+2})+\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m) \alpha(p_{n+2},p_{n+3})d(p_{n+1},p_{n+3})+\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m)\alpha(p_{n+3},p_m)d(p_{n+3},p_m)} g(t)dt$$

$$\le \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1}) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \alpha(p_j,p_m)\right) \alpha(p_i,p_{i+1})d(p_i,p_{i+1}) + \prod_{i=n+1}^{m-1} \alpha(p_i,p_m)\alpha(p_{m-1},p_m)d(p_{m-1},p_m)} g(t)dt$$

$$\le \int_0^{\alpha(p_n,p_{n+1})k^n d(p_0,p_1) + \sum_{i=n+1}^{m-2} \left(\prod_{j=n+1}^{i} \alpha(p_j,p_m)\right) \alpha(p_i,p_{i+1})k^i d(p_0,p_1)}$$

$$+\prod_{i=n+1}^{m-1} \alpha(p_i,p_m)\alpha(p_{m-1},p_m)k^{m-1}\, d(p_0,p_1) g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})k^n\, d(p_0,p_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i} \alpha(p_j,p_m)\right) \alpha(p_i,p_{i+1})k^i\, d(p_0,p_1)} g(t)dt$$

Let $S_i = \sum_{i=0}^{i}\left(\prod_{j=0}^{i} \alpha(p_j, p_m)\right) \alpha(p_i, p_{i+1})k^i\, d(p_0, p_1)$

$$\int_0^{d(p_n,p_m)} g(t)dt \leq \int_0^{d(p_0,p_1)\,[k^n\, \alpha(p_n,p_{n-1})+s_{m-1}-s_n]} g(t)dt$$

$$\leq [k^n\, \alpha(p_n, p_{n-1}) + s_{m-1} - s_n] \int_0^{d(p_0,p_1)} g(t)dt$$

Now, using the feel the $\alpha(p, q) \geq 1$ and by using the ratio test $\lim_{n\to\infty} S_n$ exits.

Hence $\{S_n\}$ is a Cauchy sequence, eventually $n, m \to \infty$ then $\int_0^{d(p_n,p_m)} g(t)dt << \lambda$.

Hence $\{p_n\}$ is a Cauchy sequence.

Let 0 is an interior of λ be given, there exists $N_1 \in N$ such that

$[k^n\, \alpha(p_n, p_{n-1}) + s_{m-1} - s_n] \int_0^{d(p_0,p_1)} g(t)dt << \lambda$ for all $m \geq N_1$ this gives

$\int_0^{d(p_0,p_1)} g(t)dt << \lambda$. For $n \geq m$, $\{p_n\}$ is a Cauchy sequence in $(S, £)$ is complete cone controlled metric space, there exists $y \in S$ such that $p_n \to y$.

There exists $N_2 \in N$ such that $\int_0^{d(p_n,y)} g(t)dt << \lambda/2$, for all $n \geq N_2$.

Hence for $n \geq N_2$ we have $\int_0^{d(p_n,y)} g(t)dt << \lambda/2$,

$$\int_0^{d(Ty,y)} g(t)dt \leq \alpha(Tp_n, Ty) \int_0^{Hd(Tp_n, Ty)} g(t)dt + \alpha(Tp_n, y) \int_0^{d(Tp_n,y)} g(t)dt$$

$$\leq \mu\, \alpha(Tp_n, Ty) \int_0^{\max\{d(p_n,y) d(p_n,p_{n+1}), d(y, Ty)\}} g(t)dt$$

$$+ \alpha(p_{n+1}, y) \int_0^{d(p_{n+1},y)} g(t)dt$$

$$\leq \mu\, \alpha(Tp_n, Ty) \int_0^{\max\{d(p_n,y), \alpha(p_n,y) d(p_n,y)+\alpha(p_{n+1},y) d(p_{n+1},y), d(y, Ty)\}} g(t)dt$$

$$+ \alpha(p_{n+1}, y) \int_0^{d(p_{n+1},y)} g(t)dt << \lambda.$$

For all $n \geq N_2, \int_0^{d(y,Ty)} g(t)dt << \lambda/\mu$ for $\mu \geq 1$ we get $(\lambda/\mu) - 1 \int_0^{d(y,Ty)} g(t)dt \in P$ and $\mu \to 0$, we get $\lambda/\mu \to 0$ and P is closed $\int_0^{-d(y,Ty)} g(t)dt \in P$ but $\int_0^{d(y,Ty)} g(t)dt \in P$.

Implies, $\int_0^{d(y,Ty)} g(t)dt = 0$ and so $y \in Ty$.

Corollary 3.4: Let $(S, £)$ be a complete cone controlled metric space and a multi-valued mapping $T : S \to CB(S)$ and $\alpha : S \times S \to [1, \infty)$ that is satisfied for every $p, q \in S$. For any $\varepsilon > 0, \int_0^{\varepsilon} g(t)dt >> 0$ in the definition of the function $g : P\cup\{0\} \to P\cup\{0\}$.

$$\int_0^{H(Tp,Tq)} g(t)dt \leq \theta_1 \int_0^{d(p,q)} g(t)dt \text{ for all } p, q \in S \text{ and } \theta_1 \in [0, 1).$$

$\lim_{n\to\infty} \alpha(p, p_n)$, $\lim_{n\to\infty} \alpha(p_n, p)$ and α satisfies the following conditions

$$\text{Sup}_{m\geq 1} \lim_{i\to\infty} \alpha(p_{i+1}, p_{i+2})\, \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+1}) < 1/k$$

where $k \in (0, 1)$. Then T has a unique fixed point in S.

Proof: The corollary proof can be obtained by taking the maximum value of $d(p, q)$ from the prior theorem.

Theorem 3.5: Let $(S, £)$ be a complete cone controlled metric space and a multi-valued mapping $T : S \to CB(S)$ and $\alpha : S \times S \to [1, \infty)$ that is satisfied for every $p, q \in S$. For any $\varepsilon > 0, \int_0^{\varepsilon} g(t)dt >> 0$ in the definition of the function $g : P\cup\{0\} \to P\cup\{0\}$.

$\int_0^{H(Tp,Tq)} g(t)dt \leq \mu \int_0^{\max\{d(p,q)d(p,Tp), d(q,Tq)d(p,Tq), d(q,Tp)\}} g(t)dt$ for all $p, q \in S$ and $\mu \in [0, 1)$. $\lim_{n\to\infty} \alpha(p, p_n)$, $\lim_{n\to\infty} \alpha(p_n, p)$ and α satisfies the following conditions.

$$\text{Sup}_{m\geq 1} \lim_{i\to\infty} \alpha(p_{i+1}, p_{i+2})\, \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+1}) < 1/k$$

where $k \in (0, 1)$. Then T has a unique fixed point in S.

Proof: For all $p_0 \in S$, $n \geq 1$, $p_1 \in Tp_0$ and $p_{n+1} \in Tp_n$

$$\int_0^{d(p_{n+1},p_n)} g(t)dt \leq \int_0^{H(Tp_n, Tp_{n-1})} g(t)dt$$

$$\leq \mu \int_0^{\max\{d(p_n,p_{n-1})d(p_n,Tp_n),d(p_{n-1},Tp_{n-1})d(p_n,Tp_{n-1}),d(p_{n-1},Tp_n)\}} g(t)dt$$

$$\leq \mu \int_0^{\max\{d(p_n,p_{n-1})d(p_n,p_{n+1}),d(p_{n-1},p_n)d(p_n,p_n),d(p_{n-1},p_{n+1})\}} g(t)dt$$

$$\leq \mu \int_0^{\max\{d(p_n,p_{n-1})d(p_n,p_{n+1}),d(p_{n-1},p_{n+1})\}} g(t)dt$$

$$\leq \mu \int_0^{\max\{d(p_n,p_{n-1}),d(p_{n-1},p_{n+1})\}} g(t)dt$$

Case (i): If $\int_0^{d(p_{n+1},p_n)} g(t)dt \leq \mu \int_0^{d(p_{n-1},p_n)} g(t)dt \leq \mu^n \int_0^{d(p_0,p_1)} g(t)dt$

For $n > m$ we have

$$\int_0^{d(p_n,p_m)} g(t)dt \leq \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)d(p_{n+1},p_m)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)\alpha(p_{n+1},p_{n+2})d(p_{n+1},p_{n+2}) +\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m)d(p_{n+1},p_m)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1})+\alpha(p_{n+1},p_m)\alpha(p_{n+1},p_{n+2})d(p_{n+1},p_{n+2})+\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m) \alpha(p_{n+2},p_{n+3})d(p_{n+1},p_{n+3})+\alpha(p_{n+1},p_m)\alpha(p_{n+2},p_m)\alpha(p_{n+3},p_m)d(p_{n+3},p_m)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})d(p_n,p_{n+1}) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(p_j,p_m)\right)\alpha(p_i,p_{i+1})d(p_i,p_{i+1}) + \prod_{i=n+1}^{m-1}\alpha(p_i,p_m)\alpha(p_{m-1},p_m)d(p_{m-1},p_m)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})k^n d(p_0,p_1) + \sum_{i=n+1}^{m-2}\left(\prod_{j=n+1}^{i}\alpha(p_j,p_m)\right)\alpha(p_i,p_{i+1})k^i d(p_0,p_1) + \prod_{i=n+1}^{m-1}\alpha(p_i,p_m)\alpha(p_{m-1},p_m)k^{m-1} d(p_0,p_1)} g(t)dt$$

$$\leq \int_0^{\alpha(p_n,p_{n+1})k^n d(p_0,p_1) + \sum_{i=n+1}^{m-1}\left(\prod_{j=n+1}^{i}\alpha(p_j,p_m)\right)\alpha(p_i,p_{i+1})k^i d(p_0,p_1)} g(t)dt$$

Let $S_i = \sum_{i=0}^{i}\left(\prod_{j=0}^{i}\alpha(p_j, p_m)\right)\alpha(p_i, p_{i+1})k^i d(p_0, p_1)$

$$\int_0^{d(p_n,p_m)} g(t)dt \leq \int_0^{d(p_0,p_1)[k^n \alpha(p_n,p_{n-1})+s_{m-1}-s_n]} g(t)dt$$

$$\leq [k^n\, \alpha(p_n, p_{n\text{-}1}) + s_{m\text{-}1} - s_n] \int_0^{d(p_0, p_1)} g(t)dt$$

Now, using the feel the $\alpha(p, q) \geq 1$ and by using the ratio test $\lim_{n\to\infty} S_n$ exits.

Hence $\{S_n\}$ is a Cauchy sequence, eventually $n, m \to \infty$ then $\int_0^{d(p_n, p_m)} g(t)dt << \lambda$.

Hence $\{p_n\}$ is a Cauchy sequence.

Let 0 is an interior of λ be given, there exists $N_1 \in N$ such that

$[k^n\, \alpha(p_n, p_{n\text{-}1}) + s_{m\text{-}1} - s_n] \int_0^{d(p_0, p_1)} g(t)dt << \lambda$ for all $m \geq N_1$ this gives

$\int_0^{d(p_0, p_1)} g(t)dt << \lambda$. For $n \geq m$, $\{p_n\}$ is a Cauchy sequence in $(S, £)$ is complete cone controlled metric space, there exists $y \in S$ such that $p_n \to y$. There exists $N_2 \in N$ such that $\int_0^{d(p_n, y)} g(t)dt << \lambda/2$, for all $n \geq N_2$.

Hence for $n \geq N_2$ we have $\int_0^{d(p_n, y)} g(t)dt << \lambda/2$,

$$\int_0^{d(Ty, y)} g(t)dt \leq \alpha(Tp_n, Ty) \int_0^{Hd(Tp_n, Ty)} g(t)dt + \alpha(Tp_n, y) \int_0^{d(Tp_n, y)} g(t)dt$$

$$\leq \mu\, \alpha(Tp_n, Ty) \int_0^{\max\{d(p_n, y) d(p_n, p_{n+1}), d(y, Ty), d(p_n, Ty), d(y, Tp_n)\}} g(t)dt$$

$$+ \alpha(p_{n+1}, y) \int_0^{d(p_{n+1}, y)} g(t)dt$$

$$\leq \mu\, \alpha(Tp_n, Ty) \int_0^{\max\{d(p_n, y), \alpha(p_n, y) d(p_n, y) + \alpha(p_{n+1}, y) d(p_{n+1}, y), d(y, Ty) d(p_n, Ty), d(y, Ty)\}} g(t)dt$$

$$+ \alpha(p_{n+1}, y) \int_0^{d(p_{n+1}, y)} g(t)dt << \lambda.$$

For all $n \geq N_2, \int_0^{d(y, Ty)} g(t)dt << \lambda/\mu$ for $\mu \geq 1$ we get $(\lambda/\mu) - 1 \int_0^{d(y, Ty)} g(t)dt \in P$ and $\mu \to 0$, we get $\lambda/\mu \to 0$ and P is closed $\int_0^{-d(y, Ty)} g(t)dt \in P$ but $\int_0^{d(y, Ty)} g(t)dt \in P$.

Implies, $\int_0^{d(y, Ty)} g(t)dt = 0$ and so $y \in Ty$.

Case (ii): $\int_0^{d(p_{n+1}, p_n)} g(t)dt \leq \mu \int_0^{d(p_{n+1}, p_{n\text{-}1})} g(t)dt$ then we get

$$\leq \mu \int_0^{\alpha(p_{n+1},p_n)d(p_{n+1},p_n)+\alpha(p_n,p_{n-1})d(p_n,p_{n-1})} g(t)dt$$

$$(1-\mu\,\alpha(p_{n+1},p_n)) \int_0^{d(p_{n+1},p_n)} g(t)dt \leq \mu\,\alpha(p_n,p_{n-1}) \int_0^{d(p_n,p_{n-1})} g(t)dt$$

$$\leq \frac{\mu\alpha(p_n,p_{n-1})}{1-\mu\alpha(p_{n+1},p_n)} \int_0^{d(p_n,p_{n-1})} g(t)dt$$

$$\leq \delta \int_0^{d(p_n,p_{n-1})} g(t)dt$$

Where $\delta = \dfrac{\mu\alpha(p_n,p_{n-1})}{1-\mu\alpha(p_{n+1},p_n)} < 1$, for $n > m$

$$\int_0^{d(Ty,y)} g(t)dt \leq \alpha(Tp_n, Ty) \int_0^{Hd(Tp_n,Ty)} g(t)dt + \alpha(Tp_n, y) \int_0^{d(Tp_n,y)} g(t)dt$$

$$\leq \mu\,\alpha(Tp_n, Ty) \int_0^{\max\{d(p_n,y)d(p_n,p_{n+1}),d(y,Ty)d(p_n,Ty),d(y,Tp_n)\}} g(t)dt$$

$$+\ \alpha(p_{n+1},y) \int_0^{d(p_{n+1},y)} g(t)dt$$

$$\leq \mu\alpha(Tp_n,Ty) \int_0^{\max\{d(p_n,y),\alpha(p_n,y)d(p_n,y)+\alpha(p_{n+1},y)d(p_{n+1},y),d(y,Ty)d(p_n,Ty),d(y,Tp_n)\}} g(t)dt$$

$$+\ \alpha(p_{n+1},y) \int_0^{d(p_{n+1},y)} g(t)dt << \lambda.$$

For all $n \geq N_2, \int_0^{d(y,Ty)} g(t)dt << \lambda/\mu$ for $\mu \geq 1$ we get $(\lambda/\mu) - 1 \int_0^{d(y,Ty)} g(t)dt \in P$ and $\mu \to 0$, we get $\lambda/\mu \to 0$ and P is closed $\int_0^{-d(y,Ty)} g(t)dt \in P$ but $\int_0^{d(y,Ty)} g(t)dt \in P$.

Implies, $\int_0^{d(y,Ty)} g(t)dt = 0$ and so $y \in Ty$.

For Uniqueness

$$\int_0^{d(y,z)} g(t)dt \leq \int_0^{Hd(Ty,Tz)} g(t)dt$$

$$\leq \lambda \int_0^{\max\{d(y,z)d(y,Ty),d(z,Tz)d(y,Tz)d(z,Ty)\}} g(t)dt$$

$$\leq \lambda \int_0^{\max\{d(y,z)d(y,y),d(z,z)d(y,z)d(z,y)\}} g(t)dt$$

$$\leq \lambda \int_0^{d(y,z)} g(t)dt$$

This implies that

$$\int_0^{d(y,z)} g(t)dt = 0.$$

Hence, T has a unique fixed point in S.

Corollary 3.6: Let (S, £) be a complete cone controlled metric space and a multi-valued mapping $T : S \to \mathrm{CB}(S)$ and $\alpha : S \times S \to [1, \infty)$ that is satisfied for every $p, q \in S$. For any $\varepsilon > 0, \int_0^{\varepsilon} g(t)dt >> 0$ in the definition of the function $g : P \cup \{0\} \to P \cup \{0\}$.

$$\int_0^{H(Tp,Tq)} g(t)dt \leq \delta \int_0^{\max\{d(p,q)d(p,Tp),d(q,Tq)\}} g(t)dt \text{ for all } p, q \in S \text{ and } \delta \in [0,1).$$

$\lim_{n\to\infty} \alpha(p, p_n)$, $\lim_{n\to\infty} \alpha(p_n, p)$ and α satisfies the following conditions

$$\mathrm{Sup}_{m\geq 1} \lim_{i\to\infty} \alpha(p_{i+1}, p_{i+2})\, \alpha(p_{i+1}, p_m) / \alpha(p_i, p_{i+1}) < 1/k$$

where $k \in (0,1)$. Then T has a unique fixed point in S.

Proof: The corollary proof is as follows right away, since

$$\int_0^{\max\{d(p,q)d(p,Tp),d(q,Tq)\}} g(t)dt \leq \int_0^{\max\{d(p,q)d(p,Tp),d(q,Tq),d(p,Tq),d(q,Tp)\}} g(t)dt.$$

Remark: Our results extend the corresponding results of Dinesh et al. [11] and others.

4. Applications:

Mathematical Biology:

In biological systems, such as population models or ecosystems, interactions between different species can be intricate and multifaceted. The study of species stability and persistence in dynamic ecosystems where growth rates and interactions are influenced by various factors is supported by fixed point results in cone controlled metric spaces. Integral-type contractions are particularly useful in modeling how populations remain stable in the face of varying environmental conditions over time.

Optimization Problems:

Many real-world optimization problems involve maximizing or minimizing a function, and they often admit multiple feasible solutions. Such systems, where a single input may correspond to several outputs, can be modeled using multivalued mappings. Cone controlled metric spaces provide a framework for extending vector space optimization, while integral-type contractions help establish convergence criteria. This ensures that optimization algorithms produce solutions that are both mathematically sound and practically applicable.

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