

The Mathematics Education
Volume-LIX, No. 4, December 2025

ISSN 0047-6269

Journal website: <https://internationaljournalsiwan.com/Mathematic.php>

ORCID Link: <https://orcid.org/0009-0006-7467-6080>

International Impact Factor: 7.75 <https://impactfactorservice.com/home/journal/2295>

Google Scholar: <https://scholar.google.com/citations?hl=en&user=UOfM8B4AAAAJ>

Refereed and Peer-Reviewed Quarterly Journal



An Observation on Direct Product of Rings

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(Received: December 1, 2025; Accepted: December 18, 2025;

Published Online: December 31, 2025)

Abstract:

This short note deals with the well known notion of direct product of rings. We report an important and useful observation pertaining to the direct product of rings.

Keywords: ring, direct product, matrix rings

1. Introduction:

The notion of direct product of rings is of fundamental importance in algebra [1]. It enables one to construct a new ring. In this note we exhibit that the direct product of two or more rings can be expressed in terms of matrices. This technique enables one to form a matrix ring using two or more given rings. This method of handling direct product through matrices is very useful in group theory also [2].

We report our observation in the next section.

[80]

2. Observation:

Let R_1 and R_2 are any two rings, then the direct product of R_1 and R_2 can be expressed as follows. $R_1 \times R_2 = \left\{ \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} : a_1 \in R_1, a_2 \in R_2 \right\}$. Clearly $R_1 \times R_2$ is a ring under the addition and the multiplication of matrices. Similarly one can express the direct product of more than two rings also.

This observation leads to the following theorem.

Theorem: Let R be the direct product of two rings R_1 and R_2 , then R is always isomorphic to a matrix ring S .

Proof: Consider a matrix ring $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in R_1, b \in R_2 \right\}$. Since R be the direct product of two rings R_1 and R_2 and therefore we can write $R = \{(a, b) : a \in R_1, b \in R_2\}$. Let us define a mapping $f : R \rightarrow S$ such that $f((a, b)) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$.

It is easy to check that;

(i) f is a surjective mapping.

(ii) Let $x = (a, b), y = (c, d) \in R$. Then $f(x + y) = (a + c, b + d) = \begin{pmatrix} a + c & 0 \\ 0 & b + d \end{pmatrix} = f(x) + f(y)$.

(iii) $f(xy) = (ac, bd) = \begin{pmatrix} ac & 0 \\ 0 & bd \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} = f(x) f(y)$.

Hence R and S are isomorphic. Therefore the direct product of two rings R_1 and R_2 is always isomorphic to a matrix ring S .

3. Examples:

It may be emphasized that this technique appears to be more intuitive when we obtain the direct product of a ring with itself. Example 2 exhibits that the ring of all 2×2 diagonal matrices over the real numbers is nothing but the direct product of the ring of real numbers with itself.

1. Let $R_1 = Z_2 = R_2$ then we have

$$R_1 \times R_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

which is a ring under the addition and the multiplication of matrices modulo two.

2. Let $R_1 = \mathfrak{R}$ be the ring of real numbers and $R_2 = R_1$. Then we have

$$R_1 \times R_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathfrak{R} \right\}$$

which is a ring under addition and multiplication of matrices. Hence the ring of all 2×2 diagonal matrices over the real numbers is nothing but the direct product of the ring of real numbers with itself.

References:

- [1] Herstein, I. N. : Topics in Algebra, Wiley India, 2011.
- [2] Pandey, S. K. : Concepts of Modern Algebra: Groups, Narosa Publishing House, New Delhi, 2024.