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The Polynomial Set $R_n^k\{(x_n), y\}$ in terms of the Sister Celin Polynomials $C_j(x_1^r)$

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Abstract :

The object of this paper is systematic study of multivariable generalized hypergeometric polynomial set $R_n^k\{(x_n), y\}$, obtained with the help of generating function involving Lauricella function in the notation of Burchanall and Chaundy [3], where n denotes the order of the polynomial set. The polynomial happens to be generalization of as many as thirty eight orthogonal and non-orthogonal polynomial set such as Legendre, Laguerre, Bateman, Jacobi, Tchebichef, Meixner, Lahiri, Charlier, Jackson, Hermite, Sylvester, etc.

These polynomials set are of at most important and wide application in Mathematical Physics, Bio-physics, flow through porous media, automatic computers, statistics and various other branches of science, engineering and technology.

The newly defined polynomial set may be of immanse use in further development of the subject.

Introduction :

In this paper an effort has been made to given some additional and more

generating relations for the generalized hypergeometric polynomial set $R_n^k\{(x_n), y\}$. Various known results have also been worked out on specializing the parameters. These results are given in articles deals with the expansion of $R_n^k\{(x_n), y\}$ in the form of finite series containing Laguerre polynomials and Lauricella functions. Many interesting new results may be obtained on specializing the respective parameters. The generalized hypergeometric polynomials set $R_n^k\{(x_n), y\}$ has been presented in a finite series containing the product of Jacobi polynomials and Lauricella functions. The generalized hypergeometric polynomial set $R_n^k\{(x_n), y\}$ has been presented in a finite series containing the product of sister celine polynomials and Lauricella functions. Our generalized hypergeomertic polynomial set has been expressed into the product of polynomials $G_i^{\lambda_1}(x_1^r, s^1 p^1)$ and Lauricella function.

A number of results as special cases have been arrived of from the general result in which some of them are believed so be new.

Proof : We achieve from

$$\sum_{n=0}^{\infty} R_n^k\{(x_n), y\} t^n = \xi^{\mu} \sum_{n=0}^{\infty} \sum_{p_1, p_2, \ldots, p_m=0}^{\infty} \frac{(\mu)_{p_1} (-\lambda_1 y^{r_1} t^{r_1})^{p_1}}{p_1!}$$

$$\times \frac{[(E_h)]_{n+p_2 \ldots . + p_m} [(A_p)]_n [(C_{u_1})]_{p_1} [(C_{u_2})]_{p_2} \ldots . [(C_{u_m})]_{p_m}}{[(F_k)]_{n+p_2 \ldots . . + p_m} [(B_q)]_n [(D_{v_1})]_{p_1} [(D_{v_2})]_{p_2} \ldots . [(D_{v_m})]_{pm}}$$

$$\times \frac{\lambda^n (\lambda_2 x_2^{r_2})^{p_2} \ldots . . (\lambda_m x_m^{r_m})^{p_m} (x_1^r)^n t^{n + r_1 p_1 + r_2 p_2 + \ldots + r_m p_m}}{n! \qquad p_2! \qquad p_m!} \quad \text{(i)}$$

$$(x_1^{r_1})^n = \frac{\left(\frac{1}{2}\right)_n (n!)^2 \left[\left(f_{g_1}^1\right)\right]_n}{\left[\left(d_{g_2}^1\right)\right]_n} \sum_{j=0}^{n} \frac{(-1)^j (2j+1) c_j (x_1^r)}{(n-j)!(n+j+1)!}$$

Hence can be written as

$$\sum_{n=0}^{\infty} R_n^k\{(x_n), y\} t^n = \sum_{n=0}^{\infty} \sum_{p_1, p_2, \ldots, p_m=0}^{\infty} \sum_{j=0}^{n} \frac{[(E_h)]_{n+p_2 \ldots . + p_m}}{[(F_k)]_{n+p_2 \ldots . . + p_m}}$$

$$\times \frac{[(A_p)]_n\,[(C_{u_1})]_{p_1}\,[(C_{u_2})]_{p_2}\ldots.[(C_{u_m})]_{p_m}\,(-\lambda_1)^{p_1}(\mu)_{p_1}\,y^{r_1p_1}}{[(B_q)]_n\,[(D_{v_1})]_{p_1}\,[(D_{v_2})]_{p_2}\ldots.[(D_{v_m})]_{p_m}\,\xi^{p_1}p_1!}$$

$$\times \frac{(\lambda_2 x_2^{r2})^{p2}.....(\lambda_m x_m^{rm})^{pm}\,\lambda^n\left(\frac{1}{2}\right)_n\,n!\,(f_{g_1}^1)_n\,(-1)^i}{p_2!\qquad p_m!\qquad [(d_{g_2}^1)]_n\qquad (n-j)!}$$

$$\times \frac{(2j+1)c_j(x_1^r)\,t^{n+r_1p_1+r_2p_2+\ldots+r_mp_m}}{(n+j+1)!}$$

$$= \xi^{\mu} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \sum_{p_1=0}^{\left[\frac{n}{r_1}\right]} \sum_{p_2=0}^{\left[\frac{n-r_1p_1}{r_2}\right]} \ldots\ldots\ldots\ldots \sum_{p_m=0}^{\left[\frac{n-r_1p_1-r_2p_2\ldots\ldots r_mp_m}{r_m}\right]}$$

$$\times \frac{[(E_h)]_{n-r_1p_1-(r_2-1)p_2-\ldots-(r_m-1)p_m}\,[(A_p)]_{n-r_1p_1-r_2p_2-\ldots-r_mp_m}}{[(F_k)]_{n-r_1p_1-(r_2-1)p_2-\ldots-(r_m-1)p_m}\,[(B_q)]_{n-r_1p_1-r_2p_2-\ldots.r_mp_m}}$$

$$\times \frac{[(C_{u_1})]_{p_1}\,[(C_{u_2})]_{p_2}\ldots[(C_{u_m})]_{p_m}\,(-\lambda_1)^{p_1}(\mu)_{p_1}\,y^{r_1p_1}(\lambda_2 x_2^{x_2})^{p_2}}{[(D_{v_1})]_{p_1}\,[(D_{v_2})]_{p_2}\ldots.[(D_{v_m})]_{pm}\qquad \xi^{p_1}\qquad p_1!\qquad p_2!}$$

$$\times \frac{\left[\left(\frac{1}{2}\right)\right]_{n-r_1p_1-r_2p_2-\ldots.-r_mp_m}\left[\left(f_{g_1}^1\right)\right]_{n-r_1p_1-r_2p_2-\ldots.-r_mp_m}(-1)^j}{\left[\left(d_{g_2}^1\right)\right]_{n-r_1p_1-r_2p_2-\ldots.-r_mp_m}(n-j-r_1p_1-r_2p_2+\ldots.+r_mp_m)!}$$

$$\times \frac{(2j+1)\lambda^{n-r_1p_1-r_2p_2-\ldots-r_mp_m}\,(n-r_1p_1-r_2p_2-\ldots.-r_mp_m)!Cj(x^{r_1})t^n}{(n+j+1-r_1p_1-r_2p_2-\ldots.-r_mp_m)!}$$

Equating the co-efficient of t^n from both sides and after little simplification, we arrive at

$$R_n^k\{(x_n),y\} = \mu\xi^{\mu} \sum_{j=0}^{n} \frac{n!\left(\frac{1}{2}\right)_n\left[\left(f_{g_1}^1\right)\right]_n\,(-1)^j\,(2j+1)c_j(x_1^{r_1})}{[(g_{g_2}^1)]_n\;(n-j)!\;\;(n+j+1)!}$$

$$\times \sum_{p_1,p_2,\ldots,p_m=0}^{\infty} \frac{[1-(F_k)-n]_{r_1p_1+(r_2-1)p_2+\ldots+(r_m-1)p_m}}{[1-(E_h)-n]_{r_1p_1+(r_2-1)p_2+\ldots+(r_m-1)p_m}}$$

$$\times \frac{[1-(B_q)-n]_{r_1p_1+r_2p_2+\ldots+r_mp_m}\ [1-(d_{g_2}^1)-n]_{r_1p_1+r_2p_2+\ldots+r_mp_m}}{[1-(A_p)-n]_{r_1p_1+r_2p_2+\ldots+r_mp_m}\,[1-(f_{g_1}^1)-n]_{r_1p_1+r_2p_2+\ldots+r_mp_m}}$$

$$\times \frac{(-n+j)_{r_1p_1-r_2p_2-\ldots-r_mp_m}\,[(C_{u_1})]_{p_1}\,[(C_{u_2})]_{p_2}\ldots..[(C_{u_m})]_{p_m}\,(\mu)_{p_1}}{[(D_{v_1})]_{p_1}\,[(D_{v_2})]_{p_2}\ldots.[(D_{v_m})]_{p_m}\qquad \xi^{p_1}\qquad p_1!}$$

$$\times \frac{(-n-j-1)_{r_1p_1-r_2p_2-\ldots-r_mp_m}\,(-\lambda_1)^{p_1}\,y^{r_1p_1}(\lambda_2x_2^{x_2})^{p_2}}{\left(\frac{1}{2}-n\right)_{r_1p_1+r_2p_2+\ldots+r_mp_m}\qquad p_2!}$$

$$\times \frac{\left[(\lambda_m x_m^{x_m})^{p_m}\,(-1)^{r_1(h+k+p+q+g_1+g_2+1)}\,(-1)^{r_2(h+k+p+q+g_1+g_2+1)+h+k}\right.}{\left.(-n)_{r_1p_1-r_2p_2-\ldots-r_mp_m}\,(-1)^{r_m(h+k+p+q+g_1+g_2+1)+h+k}\right]}{p_m!} \qquad \text{(ii)}$$

Where j is non-integer. The single terminiting factor $(-n)_{r_1p_1+r_2p_2+\ldots+r_mp_m}$ makes all summation run up to ∞ in (ii).

Corollary I: For $r_2 > 1,\ldots., r_m > 1$, we get

$$R_n^k\{(x_n), y\} = M\,\xi^{\mu} \sum_{j=0}^{n} \frac{\left(\frac{1}{2}\right)_n\,(n)!\,\left[\left(f_{g_1}^1\right)\right]_n\,(-1)^j\,(2j+1)c_j(x_1^r)}{[(d_{g_2}^1)]_n\ (n-j)!\ \ (n+j+1)!}$$

$$\times F_{h+p:\,v_1,v_2,\ldots,v_m}^{1+k+q:\,u_1,u_2,\ldots,u_m}\left[\begin{matrix}[(-n+j):\,r_1,\,r_2,..,\,r_m],\,[(-n+j-1):\,r_1,\,r_2,..,\,r_m] \\ [(-n):\,r_1,\,r_2,..,\,r_m],\,\left[\left(\frac{1}{2}-n\right):\,r_1,\,r_2,..,\,r_m\right]\end{matrix}\right.$$

$$\left.\begin{matrix}[(1-(d_{g_1}^1)-n): r_1, r_2,..., r_m], [(1-(F_k)-n): r_1, r_2-1,..., r_m-1],\\ [(1-(f_{g_2}^1)-n): r_1, r_2,..., r_m], [(1-(E_h)-n): r_1, r_2-1,..., r_m-1],\\ [(1-(B_q)-n): r_1, r_2,..., r_m]\\ [(1-(A_p)-n): r_1, r_2,..., r_m]\end{matrix}\right]$$

$$\left[\begin{matrix}[(C_{u_1}):1], [(C_{u_2}):1],..., [(C_{u_m}):1], [(\mu):1]\\ [(D_{v_1}):1], [(D_{v_2}):1],...,[(D_{v_m}):1],\end{matrix}\ \text{—}\ \frac{-\lambda_1 y^{r_1}(-1)^{r_1(h+k+p+q+1)}}{\xi\lambda^{r_1}},\right.$$

$$\left.\frac{\lambda_2 x_2^{r_2}(-1)^{r_2(h+k+p+q+1)}}{\lambda^{r_2}},...,\frac{\lambda_m x_m^{r_m}(-1)^{r_m(h+k+p+q+g_1+g_2+1)+h+k}}{\lambda^{r_m}}\right] \quad \text{(iii)}$$

Corollary II: For $r_2 > 1$ and ....., $r_m > 1$, we get

$$R_n^k\{(x_n), y\} = \frac{\mu\xi^{\mu}\, n!\left(\frac{1}{2}\right)_n\left[\left(f_{g_1}^1\right)\right]_n}{[(d_{g_2}^1)]_n}\sum_{j=0}^{n}\frac{(-1)^j\,(2j+1)c_j(x_1^r)}{(n-j)!\ \ (n+j+1)!}$$

$$\times F_{h+p:\,v_1, v_2,..., v_m}^{1+k+q:\,u_1, u_2,..., u_m}\left[\begin{matrix}[(-n+j): r_1, 1,.., r_m], [(-n-j-1): r_1, 1,.., r_m]\\ [(-n): r_1, 1,.., r_m], [\left(\frac{1}{2}-n\right): r_1, 1,.., r_m]\end{matrix}\right.$$

$$\left.\begin{matrix}[(1-(d_{g_1}^1)-n): r_1, 1,..., r_m], [(1-(F_k)-n): r_1, 0,..., r_m-1]\\ [(1-(f_{g_2}^1)-n): r_1, 1,..., r_m], [(1-(E_h)-n): r_1, 0,..., r_m-1]\\ [(1-(B_q)-n): r_1, 1,..., r_m]\\ [(1-(A_p)-n): r_1, 1,..., r_m]\end{matrix}\right]$$

$$\left[\begin{matrix}[(C_{u_1}):1], [(C_{u_2}):1],..., [(C_{u_m}):1], [(\mu):1]\\ [(D_{v_1}):1], [(D_{v_2}):1],...,[(D_{v_m}):1],\end{matrix}\ \text{—}\ \frac{-\lambda_1 y^{r_1}(-1)^{r_1(h+k+p+q+g_1+g_2+1)}}{\xi\lambda^{r_1}},\right.$$

$$\left.\frac{\lambda_2 x_2(-1)^{(2h+2k+p+q+g_1+g_2+1)}}{\lambda},...,\frac{\lambda_m x_m^{r_m}(-1)^{r_m(h+k+p+q+g_1+g_2+1)+h+k}}{\lambda^{r_m}}\right] \text{(iv)}$$

Corollary III: For $r_2 > 1$, ..., $r_m = 1$, we have

$$R_n^k\{(x_n), y\} = \frac{\mu\xi^{\mu}\, n!\left(\frac{1}{2}\right)_n \left[\left(f_{g_1}^1\right)\right]_n}{\left[(d_{g_2}^1)\right]_n} \sum_{j=0}^{n} \frac{(-1)^j\,(2j+1)c_j(x_1^r)}{(n-j)!\;(n+j+1)!}$$

$$\times F_{h+p:\, v_1, v_2,\ldots, v_m}^{1+k+q:\, u_1, u_2,\ldots, u_m}\left[\begin{matrix}[(-n+j): r_1, r_2,.., 1], [(-n-j-1): r_1, r_2,.., 1],\\ [(-n): r_1, r_2,.., 1], [\left(\frac{1}{2}-n\right): r_1, r_2,.., 1],\\ [(1-(d_{g_1}^1)-n): r_1, r_2,..., 1], [(1-(F_k)-n): r_1, r_2,..., 0],\\ [(1-(f_{g_2}^1)-n): r_1, r_2,..., 1], [(1-(E_h)-n): r_1, r_2,..., 0],\\ [(1-(B_q)-n): r_1, r_2,..., 1]\\ [(1-(A_p)-n): r_1, r_2,..., 1]\end{matrix}\right.$$

$$\left.\begin{matrix}[(C_{u_1}):1], [(C_{u_2}):1],..., [(C_{u_m}):1], [(\mu):1]\\ [(D_{v_1}):1], [(D_{v_2}):1],...,[(D_{v_m}):1],\end{matrix}\right. \frac{-\lambda_1 y^{r_1}(-1)^{r_1(h+k+p+q+g_1+g_2+1)}}{\xi\lambda^{r_1}},$$

$$\left.\frac{\lambda_2 x_2^{r_2}(-1)^{r_2(h+k+p+q+1)+h+k+g_1+g_2}}{\lambda^{r_2}},\ldots,\frac{\lambda_m x_m^{r_m}(-1)^{r_m(2h+2k+p+q+1)+g_1+g_2}}{\lambda}\right] \text{(v)}$$

Corollary IV: For $r_2 = 1 = ...= r_m$, we get

$$R_n^k\{(x_n), y\} = \frac{\mu\xi^{\mu}\, n!\left(\frac{1}{2}\right)_n \left[\left(f_{g_1}^1\right)\right]_n}{\left[(d_{g_2}^1)\right]_n} \sum_{j=0}^{n} \frac{(-1)^j\,(2j+1)c_j(x_1^r)}{(n-j)!\;(n+j+1)!}$$

$$\times F_{h+p:\, v_1, v_2,\ldots, v_m}^{1+k+q:\, u_1, u_2,\ldots, u_m}\left[\begin{matrix}[(-n+j): r_1, 1,..., 1], [(-n-j-1): r_1, 1,..., 1],\\ [(-n): r_1, 1,..., 1], [\left(\frac{1}{2}-n\right): r_1, 1,..., 1],\\ [(1-(d_{g_1}^1)-n): r_1, 1,..., 1], [(1-(F_k)-n): r_1, 0,..., 0],\\ [(1-(f_{g_2}^1)-n): r_1, 1,..., 1], [(1-(E_h)-n): r_1, 0,..., 0],\end{matrix}\right.$$

$$\left.\begin{matrix}[(1\text{-}(B_q)\text{-}n): r_1, 1,..., 1] \\ [(1\text{-}(A_p)\text{-}n): r_1, 1,..., 1]\end{matrix}\right]$$

$$\left[\begin{matrix}[(C_{u_1}):1], [(C_{u_2}):1],..., [(C_{u_m}):1], [(\mu):1] \\ [(D_{v_1}):1], [(D_{v_2}):1],...,[(D_{v_m}):1],\end{matrix}\right. \frac{-\lambda_1 y^{r_1}(-1)^{r_1(h+k+p+q+1)+g_1+g_2}}{\xi\lambda^{r_1}},$$

$$\left.\frac{\lambda_2 x_2(-1)^{2h+2k+p+q+g_1+g_2+1}}{\lambda},...,\frac{\lambda_m x_m(-1)^{2h+2k+p+q+1+g_1+g_2}}{\lambda}\right] \qquad \text{(vi)}$$

Particular Cases of (v):

For the values of the parameters given, we arrive at

$$\text{(i) } H_n(x) = \sum_{j=0}^{n} \frac{\left[\left(f_{g_1}^1\right)\right]_n \left(\frac{1}{2}\right)_n (-1)^j (2j+1)(2x)^n c_j(x^r) n!}{[(d_{g_2}^1)]_n \ (n-j)! \ (n+j+1)!}$$

$$\times F\left[\begin{matrix}-n+j, -n-j-1, -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, 1-(d_{g_2}^1)-n; \\ \\ -n; \frac{1}{2}-j, 1-(f_{g_2}^1)-n,\end{matrix}\right. \left.\begin{matrix} \\ \frac{(-1)^{g_1-g_2+1}}{x^2} \\ \\ \end{matrix}\right]$$

$$\text{(ii) } P_n(x) = \sum_{j=0}^{n} \frac{\left[\left(f_{g_1}^1\right)\right]_n \left(\frac{1}{2}\right)_n (-1)^j (2j+1) x^n c_j(x^r)}{[(d_{g_2}^1)]_n \ (n-j)! \ (n+j+1)!}$$

$$\times F\left[\begin{matrix}-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, -n+j, -n-j-1; 1-(d_{g_2}^1)-n; \\ \\ 1, \frac{1}{2}-j, 1-(f_{g_2}^1)-n,\end{matrix}\right. \left.\begin{matrix} \\ \frac{x^2-1}{x^2}(-1)^{g_1-g_2+1} \\ \\ \end{matrix}\right]$$

$$\text{(iii)}\ C_n^{\lambda}(x)=\sum_{j=0}^{n}\frac{\left[\left(f_{g_1}^{1}\right)\right]_n\left(\frac{1}{2}\right)_n(-1)^j(2j+1)(2\lambda)_n\left(\frac{x+1}{2}\right)^n c_j(x^r)}{[(d_{g_2}^{1})]_n\ (n-j)!\ (n+j+1)!}$$

$$\times F\left[\begin{matrix}-n,\frac{1}{2}-\lambda-n,-n+j-n-j-1;1-(d_{g_2}^{1})-n;\\ \qquad\qquad\qquad\qquad\qquad\qquad\frac{x-1}{x+1}(-1)^{g_1-g_2+1}\\ \lambda+\frac{1}{2},\frac{1}{2}-j,1-(f_{g_1}^{1})-n;\end{matrix}\right]$$

$$\text{(iv)}\ C_n^{\lambda}(x)=\sum_{j=0}^{n}\frac{\left[\left(f_{g_1}^{1}\right)\right]_n\left(\frac{1}{2}\right)_n(-1)^j(2j+1)(2\lambda)_n x^n c_j(x^r)}{[(d_{g_2}^{1})]_n(n-j)!\ (n+j+1)!}$$

$$\times F\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},-n+j,-n-j-1;1-(d_{g_2}^{1})-n;\\ \qquad\qquad\qquad\qquad\qquad\qquad\frac{x^2-1}{x^2}(-1)^{g_1-g_2+1}\\ \frac{1}{2}+\lambda,\frac{1}{2}-j,1-(f_{g_1}^{1})-n;\end{matrix}\right]$$

$$\text{(v)}\ \Psi_n(c,x,y)=\sum_{j=0}^{n}\frac{\left[\left(f_{g_1}^{1}\right)\right]_n\left(\frac{1}{2}\right)_n(2j+1)(-1)^{n+j}\left(\frac{1}{2}+\frac{x}{2}\right)_n x^n c_j(x_1^r)}{[(d_{g_2}^{1})]_n(c)_n(n-j)!\ (n+j+1)!}$$

$$\times F\left[\begin{matrix}-n,1-c-n,\frac{1}{2}-\frac{x}{2},-n+j,-n-j-1,1-(d_{g_2}^{1})-n;\\ \qquad\qquad\qquad\qquad\qquad\qquad y(-1)^{g_1-g_2+1}\\ c,\frac{1}{2}-\frac{x}{2}-n,\frac{1}{2}-j,1-(f_{g_1}^{1})-n;\end{matrix}\right]$$

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