The Mathematics Education **ISSN 0047-6269**
Volume - LV, No. 3, September 2021
Refereed and Peer-Reviewed Quarterly Journal
Journal website : www.internationaljournalsiwan.com

On Balanced Extension of a Set

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Abstract :

We use the idea of Convex set and convex half of a set in a linear space.

1. Introduction :

We establish a theorem concerning balanced extension of a set A as well as balanced extension of rationally convex hull of a set. Rationally convex set has been defined in our previous paper reference 3.

We also establish a theorem on product of two rationally convex sets.

2. Definition :

In linear algebra and related areas of mathematics a balanced set, circled set or disk in a vector space is a set S such that for all scalars α with $|\alpha| \leq 1$, $\alpha S \subseteq S$.

Where $\alpha S = \{ax/x \in S\}$

The balanced hull or balanced envelope for a set S is the smallest balanced set containing S.

Theorem (I.I) : Let A be a subset of a linear space L, then $B\{\text{Cr}(A)\} \subseteq \text{Cr}\{B(A)\}$.

Proof : First we show that $B(A) = \underset{\alpha}{\cup} \{\alpha A : |\alpha| \leq 1\}$

$$\text{Let } D = \underset{\alpha}{\cup} \{\alpha A : |\alpha| \leq 1\}$$

Taking $\alpha = 1, A \subseteq D$

Let $x \in D$ than $x = \alpha a$ for $a \in A$ and some α such that $|\alpha| \leq 1$.

Let β be a scalar such that $|\beta| \leq 1$.

Then $\beta x = \beta \alpha a$,

where $|\beta\alpha| = |\beta||\alpha| \leq 1$.

Therefore, $\beta x \in \beta\alpha A \subseteq D$.

Hence D is balanced.

Let S be a set such that $A \subseteq S$ and S is balanced.

Let $x \in D$ then $x = \alpha a$,

where $a \in A$, $|\alpha| \leq 1$. But $A \subseteq S \Rightarrow a \in S$. Since S is balanced. $\alpha a \in S$. Thus $x \in S$.

Hence $x \in D \Rightarrow x \in S$. Therefore $D \subseteq S$.

Hence D is the smallest balanced set containing A. So we conclude that

$$B(D) = D = \underset{\alpha}{\cup} \{\alpha A : |\alpha| \leq 1\}$$

Let z be an element of $B(\text{Cr}(A))$.

From above result

$$B(\text{Cr}(A)) = \underset{\alpha}{\cup} \{\alpha \,\text{Cr}(A) : |\alpha| \leq 1\}$$

So we see that $z \in \alpha \,\text{Cr}(A)$. For some α such that $|\alpha| \leq 1$.

Hence, we can write $z = \alpha\, l_1 a_1 + \alpha\, l_2 a_2 + \dots\dots\dots\dots + \alpha\, l_n a_n$.

Where $a_i \in A$, l_i are rational

$$l_i \geq 0 \text{ and } \sum l_i = 1$$

Hence $z \in t_1(\alpha\, a_1) + t_2(\alpha\, a_2) + \ldots\ldots\ldots\ldots + t_n(\alpha\, a_n)$

Now $a_i \in A \Rightarrow \alpha\, a_i \in B(A)$.

Therefore $z \in \mathrm{Cr}(B(A))$.

Thus $z \in B(\mathrm{Cr}(A)) \Rightarrow z \in \mathrm{Cr}(B(A))$.

Hence $B(\mathrm{Cr}(A) \subseteq \mathrm{Cr}\, B(A))$.

3. Product of two sets :

Let L be a linear space. Then $L \times L$ is a linear space in which addition and scalar multiplication are defined as follows.

$$(a, b+c, d) = (a+c, b+d) \qquad (1.1)$$

$$\alpha(a, b) - (\alpha a, \alpha b) \qquad (1.2)$$

Where $a, b, c, d \in L$ and α is a Scalar.

Theorem (I.II) : Let L be a linear space and A, B be r-convex sets of L. Let $L \times L$ be a linear space with the operations as defined by (1.1) and (1.2), above, then $A \times B$ is a r-convex set of $L \times L$.

Proof : Let (a, b) and (a^1, b^1) be elements of $A \times B$. Then $a, a^1 \in A$ and $b, b^1 \in B$.

Let α, β be rationals such that $\alpha \geq 0$, $\beta \geq 0$ and $\alpha + \beta = 1$.

Then $\alpha(a, b) + \beta(a^1, b^1) = (\alpha a, \alpha b) + (\beta a^1, \beta b^1) = (\alpha a + \beta a^1, \alpha b + \beta b^1)$.

Since $a, a^1 \in A$ and A is r-convex, so $\alpha a + \beta a^1 \in A$.

Similarly $\alpha b + \beta b^1 \in B$.

Therefore $\alpha(a, b) + \beta(a^1, b^1) \in A \times B$.

Hence $A \times B$ is a r-convex set of $L \times L$.

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