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A Static Cylindrically Symmetric Perfect Fluid Distribution in Einstein-Cartan Theory

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Abstract :

In this paper, we have discussed a static cylindrically symmetric perfect fluid distribution in Einstein-Cartan theory and have solved the field equations using suitable equation of state and by choosing a specific form for one of the metric potentials. We have assume the spins of all the individual particles composing the field to be aligned along the symmetry axis. Pressure and density have been found and the constants appearing in the solution have been found by Lichnerowicz boundary conditions.

Keywords: Symmetric perfect fluid, cylindrical, Ricci tensor, spin, Torson Tensor, Lichnerowicz boundary conditions.

1. Introduction:

In this study, we discussed Einstein-Cartan theory which attempts to incorporate the spin density of a material medium into the field equations. Spherically symmetric interior solution in Einstein-Cartan theory were reported by Kerlic [5], Kuchowicz [7-10], Parsanna [12], and Skinner and Webb [16]. Singh and Yadav [15] have also obtained static fluid spheres in Einstein-Cartan theory. Some other workers in this line are Suh [17], Banerjee [4], Arkuszewski Wiski [1], Karori et al. [18], Levi-Civita [11]. However, since in spherical symmetry it is assumed that spins are all aligned in radial direction (implying the pressure of a magnetic monopole at the center) the picture is not very physical.

Further, as a rotating system can't be spherical, naturally it seems desirable to study axi-symmetry distributions which are more physical. Keeping this fact in mind Parsanna [13] has considered the simplest axi-symmetric system namely a static cylinder of perfect fluid composed of particles having their spins aligned along the symmetric axis.

In the present paper, we have also discussed a static cylindrically symmetric perfect fluid distribution in Einstein-Cartan theory and have solved the field equations using a suitable equation of state and by choosing a specific form for one of the metric potentials. We have assumed the spins of all the individual particles composing the fluid to be aligned along the symmetry axis. Pressure and density have been found and the constants appearing in the solution have been found by Lichnerowicz boundary conditions.

2. The Field Equations:

We considered the static cylindrically symmetric metric given by

$$ds^2 = -e^{2(\mu-\nu)}(dr^2+dz^2) - r^2e^{-2\nu}d\phi^2 + e^{2\nu}dt^2 \tag{2.1}$$

where μ and ν are functions of r alone.

We have the orthonormal tetrad

$$\theta^1 = e^{\mu-\nu}dr,\ \theta^2 = re^{\nu}d\phi,\ \theta^3 = e^{\mu-\nu}dz,\ \theta^4 = e^{\nu}dt \tag{2.2}$$

The metric (2.1) now becomes

$$ds^2 = \{(\theta^1)^2 + (\theta^2)^2\,(\theta^3)^2\,(\theta^4)^2\} \tag{2.3}$$

So that, $g_{ij} = \text{diag}(-1, -1, -1, 1)$

The Einstein-Cartan field equations are

$$G_j^i = R_j^i - \frac{1}{2} R\delta_j^i = -kt_j^i \tag{2.4}$$

$$Q_{jk}^i - \delta_j^i Q_{1k}^1 - \delta_k^i Q_{j1}^1 = -KS_{jk}^i \tag{2.5}$$

Where R_{ij} is Ricci tensor, R is the scalar curvature and t_j^i is the canonical asymmetric energy momentum tensor and S_{jk}^i and Q_{jk}^i are spin and torson tensor.

The classical description of spin is defined by the relation

$$S_{jk}^i = u^i S_{jk}^i \text{ with } S_{jk} u^k = 0 \tag{2.6}$$

Where u^i is the velocity four vectors and S_{ij} is the intric angular momentum tensor.

We suppose that spin of the individual particles composing the fluid are all aligned along symmetry (Z-axis). Therefore the only none-zero components of the spin tensor S_{ij} are

$$S_{12} = -S_{21} = K \text{ (Say)} \tag{2.7}$$

$$Q_{12}^4 = -Q_{21}^4 = -kK \tag{2.8}$$

The canonical asymmetric energy momentum tensor is given by

$$t_j^i = \overline{T}_j^i + \frac{1}{2} g^{im} \Delta_k S_{jm}^k \tag{2.9}$$

$\overline{T}_j^i$ being the symmetric energy momentum tensor.

Considering the perfect fluid material distribution with anistropic pressure, the symmetric tensor $\overline{T}_j^i$ is given by

$$\overline{T}_j^i = \text{diag}(-p_r, -p_\phi, -p_z, \rho) \tag{2.10}$$

The non-zero components of the canonical tensor t_j^i are

$$t_1^1 = \overline{T}_1^1 = -p_r \tag{2.11}$$

$$t_2^2 = \overline{T}_2^2 = -p_\phi$$

$$t_3^3 = \overline{T}_3^3 = -p_z$$

$$t_4^4 = \overline{T}_4^4 = \rho$$

$$t_2^4 = \frac{1}{2} K e^{\nu-\mu} \nu'$$

$$t_4^2 = \frac{1}{2} K e^{\nu-\mu} \nu'$$

Using equations (2.4) and (2.11) the field equations may be written as (Prasanna [13])

$$e^{2(\nu-\mu)}(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2) + \frac{1}{4} k^2 K^2 = -k\rho \tag{2.12}$$

$$e^{2(\nu-\mu)}\left(\nu'^2 - \frac{\nu'}{r}\right) - \frac{1}{4} k^2 K^2 = -kp_r \tag{2.13}$$

$$e^{2(\nu-\mu)}(-\mu' - \nu'^2) - \frac{1}{4} k^2 K^2 = -kp_\phi \tag{2.14}$$

$$e^{2(\nu-\mu)}\left(\nu'^2 + \frac{\mu'}{r}\right) - \frac{1}{4} k^2 K^2 = kp_z \tag{2.15}$$

$$e^{2(\nu-\mu)}(K' + K\mu' - K\nu') = ke^{\nu-\mu}\nu' \tag{2.16}$$

$$e^{2(\nu-\mu)}(K' + K\mu' + K\nu') = ke^{\nu-\mu}\nu' \tag{2.17}$$

Adding (2.16) and (2.17), we get

$$K' + K\mu' = 0 \tag{2.18}$$

Which on integration gives

$$K = He^{-\mu} \tag{2.19}$$

where H is an arbitrary constant be determined.

The conservation equation for $j=1$ gives the continuity equation

$$\frac{dp_r}{dr} + (\rho + p_r) - (p_r - p_\phi)(\nu' - \frac{1}{r}) - (\nu' - \mu')(p_r - p_z) = -\frac{1}{2} Kk(K' + k\nu') \tag{2.20}$$

Putting $k = (-8\pi G)/c^2$ with $G = 1$, $c = 1$, we can write the field equations as

$$8\pi\rho = 16\pi^2 K^2 + e^{2(\nu-\mu)}(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2) \tag{2.21}$$

$$8\pi p_r = -8\pi p_z = 16\pi^2 K^2 + e^{2(\nu-\mu)}\left(\frac{\mu'}{r} - \nu'^2\right) \tag{2.22}$$

$$8\pi p_\phi = 16\pi^2 K^2 + e^{2(\nu-\mu)}(\mu'' + \nu'^2) \tag{2.23}$$

3. Solution of the Field Equation:

Following Hehl's approach [2, 3] by redefining pressure and density as

$$\bar{p} = p - 2\pi K^2, \bar{\rho} = \rho - 2\pi K^2 \tag{3.1}$$

The field equations reduce to

$$8\pi\bar{\rho} = e^{2(\nu-\mu)}(2\nu'' - \mu'' + \frac{2\nu'}{r} - \nu'^2) \tag{3.2}$$

$$8\pi\bar{p}_r = -8\pi\bar{p}_z = e^{2(\nu-\mu)}\left(\frac{\mu'}{r} - \nu'^2\right) \tag{3.3}$$

$$8\pi\bar{p}_\phi = e^{2(\nu-\mu)}(\mu'' + \nu'^2) \tag{3.4}$$

Also the continuity equation becomes,

$$\frac{d\bar{p}_r}{dr} + (\bar{\rho} + \bar{p}_r)\nu' - (\bar{p}_r - \bar{p}_\phi)(\nu' - \frac{1}{r}) - 2\bar{p}_r(\nu' - \mu') = 0 \tag{3.5}$$

Case 1:

We now assume an equation of state of the form

$\bar{\rho} = a\,\bar{p}_\phi$, where a is a constant.

This gives us an additional equation

$$2\nu'' + \frac{2\nu'}{r} - (1+a)\nu'^2 = (1+r)\mu'' \tag{3.6}$$

Since our set of equations are still incomplete, we will assume a particular form for one of metric potentials. For this we assume

$$\nu = \frac{B_1 r^3}{3} + B_2 \tag{3.7}$$

We can solve μ from (3.6) which we find to be

$$\mu = \frac{B_1 r^3}{2(1+a)} - \frac{B_1 r^6}{30} + C_1 r + C_2 \tag{3.8}$$

We have now four arbitrary constants B_1, B_2, C_1, C_2 which are to be determined through the boundary conditions. Assuming that the cylinder has a radius $r = r_b$, we have $r > r_b$ the field equations $R_{ij} = 0$.

A well-known solution of Einstein equations for empty space with cylindrical symmetry is the given by Levi-Civita [11] which is expressed as

$$ds^2 = -A^2 r^{2c(1-c)}(dr^2 + dz^2) - r^{2(1-c)} d\phi^2 + r^{2c} dt^2 \tag{3.9}$$

Where C and A being constants.

Since the equations (3.2) - (3.5) are similar to Einstein equations in form, we can use the Lichnerowicz boundary conditions namely that the metric potentials are C_1 across the surface $r = r_b$. Thus the continuity of μ, μ' and ν, ν' gives us

$$B_1 = \frac{C}{r_b^3}, \quad B_2 = C \log r_b - \frac{C}{3} \tag{3.10}$$

$$C_1 = \frac{6C}{r_b(1+a)} \left\{ \frac{C(1+a)}{5} - \frac{1}{2} \right\}$$

$$C_2 = \log A + C^2 \log r_b + \frac{2C}{(1+a)} - \frac{23C^2}{30}$$

Here we have for the interior of the cylinder, the solution,

$$\mu = \frac{2C}{1+a} \left\{ \frac{R}{2}(R^2 - 1) - (R - 1) \right\} - \frac{C^2}{5} \left\{ \frac{R^2}{6} - 6R \right\}$$

$$+ \log A + C^2 \log r_b - \frac{23C^2}{30} \tag{3.11}$$

$$\nu = \frac{C}{3}(R^3 - 1) + C \log r_b \; ; R = \frac{r}{r_b} \tag{3.12}$$

With the pressure and density given by

$$8\pi p_r = 16\pi^2 B^2 e^{-2\mu} + e^{2(\nu-\mu)} \left\{ \frac{6C}{2(1+a)r_b r}(R^2 - 1) + \frac{6C^2}{5r_b r}(1 - R^5) \right\}$$

$$8\pi p_z = 16\pi^2 B^2 e^{-2\mu} - e^{2(\nu-\mu)} \left\{ \frac{6C}{2(1+a)r_b r}(R^2 - 1) + \frac{6C^2}{5r_b r}(1 - R^5) \right\}$$

$$8\pi p_\phi = 16\pi^2 B^2 e^{-2\mu} + e^{2(\nu-\mu)} \left\{ \frac{6C}{(1+a)r_b^2} R \right\}$$

$$8\pi \rho = 16\pi^2 B^2 e^{-2\mu} + e^{2(\nu-\mu)} \left\{ \frac{4Ca}{(1+a)r_b^2} R \right\}$$

Case 2:

Here we choose

$\overline{\rho} = a\,\overline{p}_r$, where a is a constant.

This gives an additional equation

$$\nu'' + \frac{2\nu'}{r} - (1\text{-}a)\nu'^2 = \frac{a\mu'}{r} + \mu'' \tag{3.13}$$

Since our set of equations are still incomplete, we will take a judicious of one of the metric potentials. For this we choose

$$\nu = B_1 r^3 + B_2 \tag{3.14}$$

Where B_1 and B_2 are constants.

With this value of ν equation (3.13) yields

$$\frac{d^2\mu}{dr^2} + \frac{a}{r}\frac{d\mu}{dr} = 8B_1 r - (1\text{-}a)4B_1^2 r^4 \tag{3.15}$$

Putting $\frac{d\mu}{dr} = p$, we get

$$\frac{dp}{dr} + \frac{a}{r}p = 8B_1 r - 4(1\text{-}a)B_1^2 r^4 \tag{3.16}$$

Equation (3.16) is a linear equation in p and r. Its solution is given by

$$p = \frac{d\mu}{dr} = C_1 r^{-a} + \frac{8B_1 r^2}{2+a} - \frac{4(1\text{-}a)}{a+5} B_1^2 r^5 \tag{3.17}$$

Where C_1 is constant of integration.

Integration of (3.17) yields

$$\mu = \frac{C_1 r^{1\text{-}a}}{1\text{-}a} + \frac{8B_1^2 r^3}{3(2+a)} - \frac{2(1\text{-}a)}{3(a+5)} B_1^2 r^6 + C_2 \tag{3.18}$$

Where C_2 is another constant of integration.

As in case 1, using boundary conditions we can find the constants B_1, B_2, C_1, C_2 and also pressure and density can be written similarly.

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