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# On Characterization of Lattice *X*

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**Abstract:**

*In this article our main aim is to introduce Lattice X. We have modified the maximal inequalities for Fefferman and Stein. We have shown here that UMD condition for lattice X can be extrapolated to a Banach lattice X*. We have shown that some statements for lattice X are equivalent.*

**Keywords:** Lattice *X*, Maximal, inequalities, UMD, etc.

## 1.1 Introduction of a Lattice:

A lattice is an abstract structure studied in the mathematical sub-disciplines of order theory and abstract algebra. It consists of a partially ordered set in which every pair of elements has a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet).

## 1.2 Definition:

A lattice can be defined either order theoretically as a partially ordered set, or as an algebraic structure.

**As Partially Ordered Set:**

A partially ordered set (Poset) $(L, \leq)$ is called a lattice if it is both a join and a meet semi lattice i.e. each two elements subset $\{a, b\} \subseteq L$ has a join (i.e. least upper bound, denoted by $a \vee b$ and dually a meet (i.e. greatest lower bound denoted by $a \vee b$ and dually a meet (i.e. greatest lower bound denoted by $a \wedge b$). This definition makes $\wedge$ and $\vee$ binary operations. Both operations are monotone with respect to the given order $a_1 \leq a_2$ and $b_1 \leq b_2$ implies that $a_1 \vee b_1 \leq a_2 \vee b_2$ and $a_1 \wedge b_1 \leq a_2 \wedge b_2$.

**As Algebraic Structure:**

A lattice is an algebraic structure $(L, \vee, \wedge)$ consisting of a set $L$ and two binary commutative and associative operations $\vee$ and $\wedge$ on $L$ satisfying the following axiomatic identities for all elements $a, b \in L$ (sometimes called absorptions laws):

$$a \vee (a \wedge b) = a$$

$$a \wedge (a \vee b) = a$$

Also $\left.\begin{array}{l} a \vee a = a \\ a \wedge a = a \end{array}\right\}$ Idempotent laws.

These axioms assert that both $(L, \vee)$ and $(L, \wedge)$ are semi lattices.

**1.3 Theorem:** For a lattice $X$, the following statements are equivalent

i) $x \in$ UMD

ii) $C_p^{-1} \| f \| L_x^p \leq 1 \leq \| Sf \| L^p \leq C_p \| f \| L_x^p$ for all $f \in L_x^p$, $(1 \leq p < \infty)$

iii) $M$ is bounded in $L_x^p$ and in $L_x^{p(*)}$, $(1 < p < \infty)$

We show that (ii) and (iii) hold for all $1 < p < \infty$. We consider square functions defined in terms of fourier series. We show here that when $X = l^q$ with $1 \leq q < \infty$ (iii) reduces to the form

$$\left\| \left( \sum_{j=1}^{\infty} (Mf_j)^q \right)^{1/q} \right\|_p \leq C_{p,q} \left\| \left( \sum_{j=1}^{\infty} |f_j|^q \right)^{1/q} \right\|_p \qquad \text{(i)}$$

We modify here the maximal inequalities for Fefferman and Stein. We show that (iii) holds for $X$ and for its dual $X^*$. Here $M$ is trivially bounded $L_x^p$ if $X = c_0 \in$ UMD. The main problem here is to show equivalence. We consider here it into following steps:

**Step I:** (i) $\Rightarrow$ (ii) : Khintchin's inequalities may be written in the form

$$Sf(t, w) = \left( \sum_n | f_n(t, w)|^2 \right)^{1/2} \le C \int \left| \sum_n e_n f_n(t, w) \right| de \qquad \text{(ii)}$$

By suitable application of the (UMD)-condition, we get

$$\| Sf\| L_x^p \le c \int \left\| \sum_n e_n f_n \right\| L_x^p \, de < C_p \| f \| L_x^p \qquad \text{(iii)}$$

We now prove the converse inequality for which given $p$ and $q$ with $p \le q < \infty$. $X$ is $q$-concave (such a $q$ exists because $X$ is super reflexive), which implies

$$\left( \int \| x_e \|_x^q \, de \right)^{1/q} \le C^q(X) \left\| \left( \int |x|^q \, de \right)^{1/q} \right\|_x \qquad \text{(iv)}$$

We apply this to $x = \sum_n e_n f_n(t)$ and using Holder's inequality, we get

$$\left( \int \left\| \sum_n e_n f_n(t) \right\|_x^p d \right)^{1/q} \le C^q(X) \left( \int \left\| \sum_n e_n f_n(t)^q d \right\|^{1/q} \right)_x \le C^q(X) \, C_q^1 \, \| Sf(t) \|_x \qquad \text{(v)}$$

Where $C_q^1$ is the constant in Khinchin's inequality. Hence, for every choice of signs $e = (e_n)$, we know by the (UMD) = condition that

$\left\| \sum_n e_n f_n(t) \right\| L_x^p \le C_p(x) \| f \| L_x^{p'}$. Hence, the equivalence follows.

We now show that (i) $\Rightarrow$ (iii).

**Step 2:** Let $H$ be the $X^*$ = valued Hilbert transform, and let us denote by $B_{p1} = B_p$ $(X^*)$ its norm as an operator in $L_{x^*}^{p'}(T)$. To each $g \in L_{x^*}^{p'}$, we associate another function $G \in L_{x^*}^{p'}$ we define $(g_n)_{n>0}$ by considering

$$g_0 = g, \, g_{n+1}(t) = | \overline{H} g_n(t) |$$

$$\text{then } G(t) = \left( \sum_{n=0}^{\infty} (2B)_{p^x} \right)^{-n} g_n(t).$$

The following properties hold for the given function

$$|g(t)| \le G(t) \tag{vi}$$

$$\| G \| L_{x*}^{p''} \le 2 \| g \| L_{x*}^{p'} \tag{vii}$$

$$|HG(t)| \le 2Bp'\, G(t), \text{ a.e. } t \in T \tag{viii}$$

Now, given $f \in L_x^p$,

$$\| M_d f \| L_x^p = \sup_g \iint M_d f(.,\omega)(t) g(t, \omega) d\omega dt \tag{ix}$$

$$\le \sup_g \iint M_d f(.,\omega)(t) G(t, \omega) d\omega dt$$

where the "sup" is taken for all $g \in L_{X*}^{P'}$ of unit norm. Again, we get $| HC(.,\omega) < 2B_{p'}G(.,\omega)$, a.e., for each fixed $\omega \in \Omega$ and implies $G(.,\omega) \in A_2 \subset A_\infty$ uniformly in $\omega$. Then taking integral, we obtain

$$\int_T M_d f(.,\omega)(g) G(t, \omega) dt \le C \int_T Sf(.,\omega)(t) G(t, \omega) dt \tag{x}$$

$$\left\| \widetilde{M}_{df} \right\| L_X^P \le C \sup_g \left\| \widetilde{S} f \right\| L_X^P \| G \| L_{X*}^{P'} \le 2C \left\| \widetilde{S} f \right\| L_X^P \le C_p \| f \| L_X^P$$

Now $Mf(t) \le c\, f\, T_M\, d^f s\ (t + s) ds$, where $f_s(t) = f(t - s)$. The same relation holds between $\widetilde{M}_d$ and $\widetilde{M}$. Thus, we have shown that $\widetilde{M}$ is bounded in $L_X^P$ for all $1 \le p < \infty$, and the same holds for $X^*$ because $X^* \in$ UMD.

**Step III:** (iii) $\Rightarrow$ (i): To each $g \in L_{X*}^{P'}$, we associate $G$ and the operator $\widetilde{M}$ instead of $\widetilde{H}$, so that (vi) and (vii) hold, and (viii) is replaced by

$$G(.,\omega) \in A_1 \subset A_\infty \text{ uniformly in } \omega \tag{xi}$$

We further consider an inequality

$$\left\| \widetilde{H} f \right\| L_X^P \le \operatorname{Sup}_g \iint | Hf(.,\omega)(t) |\, G(t, \omega) d\omega dt$$

$$\le \operatorname{Sup}_g \iint | Mf(.,\omega)(t) |\, G(t, \omega) dt d\omega \le 2 \| \widetilde{M} \| L_X^P$$

$$\le C_p \| f \| L_X^P \tag{xii}$$

**1.4 Theorem:** Let $X$ be a UMD Lattice. Then, there exists $\in > 0$ such that $X^a \in$ UMD for all $0 < a < 1 + \in$.

**Proof :** We prove it into two different steps. The statement corresponding to the value of $a \leq 1$. Since $\left[L^{\infty}, X\right]_{\theta} \in$ UMD for every UMD lattice and all $0 < \theta < 1$.

Hence $\widetilde{H}f(t, \omega)^2 = f(t, \omega)^2 + 2\widetilde{H}(fHf)(t, \omega)$ of $X$-valued Hilbert transform $H$.

We further obtain the following relations

$$\left\| \widetilde{H}f \right\|^2 L^2\left(x^{\frac{1}{2}} + \int \left\| \widetilde{H} \,|\, f(t) \,|^2 \right\|^2 x^* \, dt\right)$$

$$\leq 2 \int \left( \left\| \,|\, f(t) \,|^2 \, \right\|_x + 2 \left\| H(f . \widetilde{H} f)(t) \right\|_x^2 dt \right)$$

$$\leq \left\| f \right\|^2_{L^2\left(x\frac{1}{2}\right)} + 4C(X) \int \left\| f(t)\widetilde{H}f(t) \right\|_x^2 dt$$

$$\leq 2 \left\| f \right\|^2_{L^2\left(x\frac{1}{2}\right)} + 4C(X) \left\| f \right\|^2_{L^2\left(x\frac{1}{2}\right)} \left\| \widetilde{H}f \right\|^2_{L^2\left(x\frac{1}{2}\right)} \qquad \text{(xiii)}$$

and it follows that $X^{1/2} \in$ UMD, by interaction, we obtain the same result for $X^2$ if $a = 2^{-k}$, $k \in N$.

The statement for $-1 \leq a \leq 1$ is interesting and says that the UMD condition for a lattice $X$ can be extrapolated to a Banach lattice $X^a$.

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