The Mathematics Education **ISSN 0047-6269**
Volume - LVIII, No. 4, December 2024
Journal website: https://internationaljournalsiwan.com
ORCID Link: https://orcid.org/0009-0006-7467-6080
International Impact Factor: 6.625 https://impactfactorservice.com/home/journal/2295
Google Scholar: https://scholar.google.com/citations?hl=en&user=UOfM8B4AAAAJ
Refereed and Peer-Reviewed Quarterly Journal

On Application of Gaussian Measure

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(Received: November 28, 2024; Accepted: December 21, 2024; Published Online: December 30, 2024)

Abstract:

We show here that De Finetti's theorem for exchangeable probabilities in standard spaces which are isomorphic to Borel subsets of the unit interval implies that the results due to Dubins and Freedman may not be true for separable, metrizable spaces.

The compact Hausdorff space S also admits the Borel σ - field B which is larger than the Baire σ - field B when S is not metrizable. We define $\bar{B}$ to be defined as the smallest σ - field relative to which all lower semicontinuous functions are measurable. A probability p on B has a unique regular extension p to B. We to show here that the formula $P = \int^{\infty} p\mu(dp)$ may be extended regularly from the Borel to the Borel σ - fields.

Let f_0 be a net of continuous functions, with $f_n \uparrow f$. Then $p(f) = \lim_{\alpha} p(f_0)$, the value p(f) does not depend on which net of continuous functions is used to approximate the lower semicontinuous function f. The extension of p from the lower semicontinuous functions to all Borel functions follows by the usual

procedure. We point out here that the Borel sets usually do not turn up in the measure-theoretic completion of B. Let $p \in S^*$ *is a Baire probability, defined only on B: its regular extension to B is p.*

1.1 Definition:

Let f be a non-negative Borel function on S. Then $p \to \bar{P}(f)$ is a Borel function on S^*. Let us suppose f to be lower semicontinuous. We choose a net f_α of continuous functions with $f_\alpha | f$. Now $p \to P(f_\alpha)$ is a continuous of p, and this net increases to $p \to p(f)$. The function is lower semicontinuous, and hence Borel. We consider the class of Borel f_α for which f for which $p \to \bar{p}(0 \vee f \wedge k)$ is Borel. This class includes the lower semicontinuous functions and is closed under sequential limits which comprise all Borel $f's$. Letting $k \uparrow \infty$, we get the conditions:

(i) The map $\to \bar{p}^\infty$ is weak * continuous from S^* to $(S^\infty)^*$.

(ii) Let B^∞ be the Borel σ-field of S^∞. Usually, B^∞ is larger than B^∞.

(iii) For $P \in S^*$, let $\tilde{p}^\infty$ be the regular Borel extension of p^2 to B^∞. Let f be a non-negative Borel function on S^∞. Then $\tilde{p}^\infty \to \tilde{p}^\infty(f)$ is a Borel function on S^*.

1.2 Theorem: Let S be compact Hausdorff, and P an exchangeable Baire probability on (S^∞, B^∞). Hence there exist a unique probability μ on $(S^\infty \in B^\infty)$ such that

$$P(A) = \int (S^*, B^*)\, p^\infty(A)\, \mu(dp) \text{ for all } A \in B^\infty \tag{i}$$

This representation extends regular to the Borel σ-field:

$$\bar{P}(A) = \int (S^*, \tilde{B}_0)\, \tilde{P}^\infty(A)\, \mu(dp) \text{ for all } A \in \tilde{B}^\infty \tag{ii}$$

Here the space S^* of probabilities p on (S, B) has been equipped with the Baire σ-field B^*. Let the space S^* is the same, and is equipped with the larger Borel σ-field B^*.

Proof: Let f be a non-negative lower-semi continuous function on S^∞. We show here that

$$P(f) = f(S^*, B_0)\, p^\infty(f)\, \mu(dp) \tag{iii}$$

Suppose the net f_α of continuous functions on S^∞ increases to f. Then $p \to p^\infty(f_\alpha)$ is a net of continuous functions on S^* increasing to $p \to p^\infty(f)$. The function is lower-semi continuous on S^*, we thus obtain

$$\begin{aligned} \widetilde{P}(f) &= \lim_\alpha P(f_\alpha) \\ &= \lim_\alpha \int P^\infty(f_\alpha)\, \mu(dp) \\ &= \int P^\infty(f)\, \widetilde{\mu}(dp) \qquad \text{(iv)} \end{aligned}$$

But the relation (iii) can be extended to all non-negative Borel functions.

We now consider nontopological situations. Let (S, B) be an abstract measurable space. Let S^* be the set of probabilities on B, with the weak * σ - field B^* generated by the function $p \to p(A)$ as A ranges over B. Hewitt and Savage has shown that B presentable if, for every exchangeable P on (S^*, B^∞) there is a probability μ on (S^*, B^*) with $P = P\mu$. In this terminology, from assertion of theorem we find that the Baire σ-field of a compact Hausdorff space S is presentable. It is not known whether the Borel σ-field of S is presentable deals with regular probabilities. It is also not known if the Borel σ-field of a locally convex topological vector space is presentable.

We show here that if a σ-field B of subsets of S is presentable, so is the universal completion $\widetilde{B}$. We define $\widetilde{B}$ to be the σ - field of subsets of S which are θ measurable for any probability θ on B. Hence θ has a unique extension θ to B.

1.3 Theorem: If B is a presentable σ-field of subsets of S, and $\sum$ is a σ-field of subsets of S with $B \subset \sum \subset B$, then $\sum$ is presentable.

Proof: We know that S^* is the set of probabilities on (S, B) endowed with the σ-field B^* generated by the functions $p \to p(A)$ as A ranges over B. Let B^* be the universal completion of B^*, and B^* the universal completion of B^∞. If $A \in B^\infty$, we claim that $p \to p^\infty(\widetilde{A})$ is $\widetilde{B}^*$ measurable on S^*. Hence let μ is a probability on B^*. We choose A_0 and A_1 in B^∞ with $A_0 \subset A \subset A_1$ and $P_\mu(A_0) = P_\mu(A_1)$. We thus get

$$P^\infty(A_0)\ \ P \le \bar{P}^\infty(A) \le P^\infty(A_1) \text{ for all } P = P \in S^* \qquad \text{(v)}$$

And

$$P(A_0) = P(A_1) \text{ for } \mu\text{-almost all } \in S^* \quad \text{(vi)}$$

So $p \to \overline{p}^{\infty}(A)$ is μ-measurable.

Suppose P is exchangeable on (S^*, B^{∞}). Since B is presentable by assumption, there is a probability μ on (S^*, B^*) with

$$P(A) = \int (S^*, B)\, P^{\infty}(A)\, \mu(dp) \text{ for all } A \in B^{\infty} \quad \text{(vii)}$$

We further prove that

$$\overline{P}(A) = \int (S^*, \overline{B}_0)\, \overline{P}^{\infty}(A)\, \overline{\mu}(dp) \text{ for all } A \in \overline{B}^{\infty} \quad \text{(viii)}$$

Let A_0 and A_1 be in B^{∞} with $A_0 \subset A \subset A_1$ and $P(A_0) = P(A_1)$. We take help of assumption (vii) and find that

$$P(A_0) = \int P^{\infty}(A_0)\, \mu(dp) \leq \overline{p}^{\infty}(A)\, \mu(dp)$$

$$\leq \int p(A_1)\, \mu(dp) = p(A_1) = p(A_0) \quad \text{(ix)}$$

Hence, equality holds and Theorem is proved.

We further prove that $B^{\infty} \subset \overline{B}^{\infty}$. Let $A = A_1 \times .... \times A_n \times S \times S \times ....$ with $A_i \in \tilde{B}$.

Let Q be a probability on B^{∞}, with i^{th} marginal Q_1. Because $A_i \in B$, there are sets B_i and C_i in B with $B_i \subset A_i \subset C_i$ and $Q_i\,(C_i - B_i) = 0$.

Let $B = B_i \times ..... \times B_n \times S \times S \times ....$ $C = C_1 \times ..... \times C_n \times S \times S \times ......$ Then B and C are in B^*, and $B \subset A \subset C$, it show that $Q(C - B) = 0$. Hence $A \in B^{\infty}$. Then $B^{\infty} \subset B^{\infty}$ suppose $B \subset \Sigma \subset \overline{B}$, and Q is an exchangeable probability on Σ^{∞}. Let P be the restriction of Q to B^{∞}. We find that P is exchangeable, and $P(A) = Q(A)$ for $A \in \Sigma^{\infty}$. We confine A to Σ^{∞} and show that $P^{\infty}(A) = (P\, p^{\infty})\,(A)$. We thus obtain

$$Q(A) = P(A) = \int (S^*, B_0)\, p^{\infty}(A)\, \mu(dp) \quad \text{(ix)}$$

Let $P \in S^*$ be a probability on (S, B) which extends to a probability p on (S, B). In the integral, the space S^* has been equipped with the universal completion B^* of B^*. Let W^* be the space of probabilities q on (S, Σ), equipped with the weak* σ-field Σ^*. For $p \in S^*$, let p be the restriction of p from B to Σ. In particular, $P^{\infty}(A) = (P\, p^{\infty})\,(A)$ for $A \in \Sigma^{\infty}$. By suitable application of (ix), we get

$$Q(A) = \int (S^0, \bar{B}^0)(Pp^\infty)\,(A)\,\mu(dp) \tag{x}$$

We further verify that $p \to P$ is a (B^*, Σ^*) measurable map of S^* to w^* : let $\nu = \bar{\mu}\, p^{-1}$.

Hence $Q(A) = \int (w^*, \Sigma^0)\; q^\infty(A)\; \nu(d\,q)$.

We have considered only countably additive probabilities. There are some more results involving finitely additive probabilities. Hewitt and Savage showed that any finitely additive probability mixture of finitely additive power probabilities. It may be derived from the case of finite S by a compactness argument.

By an appropriate application of assertion of theorem it leads to a different kind of representation in terms of finitely additive mixture of countably additive powers. We state this variant of theorem, let S be a set and B a σ-field of subsets of S. Let $S^*\,d$ be the set of those countably additive probabilities p on B which have finite support. In other terms, $p \in S^*\,d$ iff for some positive integer n and points $S_1,\ldots\ldots, S_n$ in S and non-negative weights with $W_1,\ldots\ldots, W_n$ with $W_1 + \ldots\ldots + W_n = 1$, we have $P(A) = \sum W_1(S_i)$. Let $B^*\,d$ be the σ-field of subsets of $S^*\,d$ spanned by the factors $p \to p(A)$ as A ranges over B. If B is a separable σ-field, then $S^*\,d \subset B^*$.

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