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Application of some notations in Topological Spaces

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Abstract :

We have introduced some notations in topological spaces and with its help conditions for continuity, semi-continuity, semi-open sets, semi-closed sets, etc. have been deduced.

Keywords : semi-open sets, semi-closed sets, semi-continuity, α-closed set

I. Introduction :

In [1], a semi-open set in a topological space has been defined by Norman Levine. In [2], a semi-closed set has been introduced by Crossley and Hildebrand. In [3], α-closed set has been defined by Njastad.

II. Definitions and preliminaries :

In context of topological spaces, we have introduced some new notation. Generally interior of a set A in a topological space X is denoted by $\text{int}(A)$ or A^0. Let us write

$$i(A) = \text{int}(A)$$

In this new notation, same standard results are as follows :

$$i(A) \subseteq A,\ i(X) = X,\ i(\phi) = \phi$$

$$i^2(A) = i(i(A)) = i(A). \text{ So we can write } i^2 = i$$

$$i(A \cap B) = i(A) \cap i(B)$$

$$i(A) \cup i(B) \subseteq i(A \cup B)$$

$$A \subseteq B \Rightarrow i(A) \subseteq i(B)$$

A set A is open $\Leftrightarrow i(A) = A$

Usually we denote complement of a set A by A^c or A'.

Here we use $C_0 A$ for complement of a set A.

So we have the following results :

$$A \cap C_0 A = \phi,\ A \cup C_0 A = X$$

$$C_0^2(A) = C_0(C_0 A) = A$$

So let $C_0^2 = I$, Identity set function.

In standard topology book relation between closure of a set denoted by $\bar{A}$ and interior of A, denoted by int A is given by

$$\bar{A} = (\text{int } A')'$$

We write $C(A)$ in place of $\bar{A}$ in our new notation.

So above result becomes

$$C(A) = C_0(i(C_0(A))) = C_0\, i\, C_0(A)$$

Hence we write

$$C = C_0\, i\, C_0$$

Thus A is closed $\Leftrightarrow C(A) = A$

For two sets A and B,

$$C(A \cup B) = C(A) \cup C(B)$$

$$C(X) = X,\ C(\Phi) = \Phi$$

$$C(C(A)) = C(A). \text{ Hence } C^2 = C$$

$$A \subseteq B \Rightarrow C(A) \subseteq C(B)$$

When $A \subseteq \text{int}\,\bar{A}$, A is defined to be pre-open.

So in our notations,

$$A \subseteq iC(A)$$

When $\overline{\text{int}(A)} \subseteq A$, A is defined to be pre-closed.

Hence in our new notations,

$$Ci(A) \subseteq A$$

As defined by Norman Levine in [1], A is semi-open if $A \subseteq Ci(A)$.

As defined by Crossley and Hildebrand in [2], A is semi-closed if $iC(A) \subseteq A$.

III. Main Results :

In terms of interior, we express conditions of continuity.

Let $g : X \to Y$, where X and Y are topological spaces and g is a continuous mapping of X into Y.

Let A be a set in Y. Then iA is open in Y. Since g is continuous, $g^{-1}(iA)$ is open in X.

Hence

$$ig^{-1}(iA) = g^{-1}(iA) \Rightarrow (ig^{-1}i)(A) = (g^{-1}i)(A)$$

But A is any set in Y, hence

$$ig^{-1}i = g^{-1}i$$

Consider the converse. So let g be a mapping such that

$$ig^{-1}i = g^{-1}i$$

So we show that g is continuous.

Let A be open in Y.

Hence $iA = A$

$$\text{So } g^{-1}(A) = g^{-1}(iA) = (g^{-1}i)A = (ig^{-1}i)(A) = ig^{-1}(iA) = ig^{-1}(A)$$

This shows that $g^{-1}(A)$ is open in X.

Hence g is continuous.

So, g is continuous $\Leftrightarrow ig^{-1}i = g^{-1}i$

Next consider the case when g is an open mapping.

Let $A \subseteq X$, then iA is open in X.

Hence $g(iA)$ is open in Y. This means

$$ig(iA) = g(iA)$$

We can write

$$(igi)(A) = g(iA) = (gi)(A)$$

But A is any set, hence

$$igi = gi$$

Conversely, let $igi = gi$

Let A be open in X i.e. $iA = A$

Hence

$$igi(A) = gi(A) \Rightarrow ig(A) = g(A)$$

So $g(A)$ is open in Y.

Hence g is an open mapping.

Therefore

$$g \text{ is an open mapping} \Leftrightarrow igi = gi$$

Corollary 1 : If f is one-one, onto and $fi = if$ then f is a homeomorphism.

Proof : Since f is one-one and onto, it is invertible. So f^{-1} exists such that ff^{-1} or $f^{-1}f$ are identically mappings.

Also

$$fi = if \Rightarrow i(fi) = i(if) = i^2 f = if = fi$$

Thus $\quad ifi = fi$

Hence f is an open mapping.

Moreover,

$$fi = if \Rightarrow f^{-1}(fi) = f^{-1}(if)$$

$$\Rightarrow \quad (f^{-1}f)(i) = f^{-1}(if) \Rightarrow ii = f^{-1}if$$

$$\Rightarrow \quad i = f^{-1}if$$

$$\Rightarrow \quad i(f^{-1}i) = (f^{-1}if)(f^{-1}i) = f^{-1}i(ff^{-1})i = f^{-1}iii = f^{-1}i$$

Thus $\quad if^{-1}i = f^{-1}i$

Hence f is continuous. Since f is open also, it is a homeomorphism.

Corollary 2 : If f is a homeomorphism, then

$$fi = if \Leftrightarrow fC = Cf$$

(i.e. i and C can be interchanged)

Proof : We know that in case f is a homeomorphism, f is one-to-one, onto and $fC_0 = C_0 f$ as mentioned in [4].

Hence

$$fi = if$$

$$\Leftrightarrow \quad C_0(fi)C_0 = C_0(if)C_0$$

$$\Leftrightarrow \quad (C_0 f)iC_0 = C_0 i(fC_0)$$

$$\Leftrightarrow \quad (fC_0)iC_0 = C_0 i(C_0 f) \text{ ; using } fC_0 = C_0 f$$

$$\Leftrightarrow \quad f(C_0 iC_0) = (C_0 iC_0)f$$

$$\Leftrightarrow \quad fC = Cf$$

Now we show some well known results in topology in terms of our new notations.

1. Complement of open set is closed.

Proof : Let A be an open set, then $iA = A$

We need to prove that $C_0 A$ is closed i.e.

$$CC_0 A = C_0 A$$

Now $CC_0 A = CC_0 iA = (C_0 iC_0)C_0 iA = C_0 i(C_0 C_0)iA = C_0 iiA = C_0 iA = C_0 A$

Hence the proof.

2. Complement of closed set is open.

Proof : Let A be closed. Then $CA = A$

We need to show that $C_0 A$ is open i.e.

$$iC_0 A = C_0 A$$

Now $iC_0 A = (C_0 C_0)iC_0 A = C_0(C_0 iC_0)A = C_0 CA = C_0 A$

Hence the proof.

3. If X, Y, Z are topological spaces $f : X \to Y$ and $g : Y \to Z$ are continuous mappings, then $gf : X \to Z$ is continuous.

Proof : Since f and g are continuous

$$if^{-1}i = f^{-1}i \text{ and } ig^{-1}i = g^{-1}i$$

Now,

$$(if^{-1}i)(ig^{-1}i) = if^{-1}(ii)g^{-1}i = if^{-1}(ig^{-1}i)$$

$$= if^{-1}(g^{-1}i) = i(f^{-1}g^{-1})i = i(gf)^{-1}i \tag{1}$$

Also

$$(if^{-1}i)(ig^{-1}i) = (f^{-1}i)(g^{-1}i) = f^{-1}(ig^{-1}i)$$

$$= f^{-1}(g^{-1}i) = (f^{-1}g^{-1})i = (gf)^{-1}i \tag{2}$$

From (1) and (2),

$$i(gf)^{-1}i = (gf)^{-1}i$$

This shows that gf is continuous.

4. Condition for continuity in terms of closure.

We have seen that f is continuous if and only if

$$if^{-1}i = f^{-1}i$$

$$\Leftrightarrow \quad (if^{-1}i)C_0 = (f^{-1}i)C_0 \Leftrightarrow C_0(if^{-1}iC_0) = C_0(f^{-1}iC_0)$$

$$\Leftrightarrow \quad C_0 i(f^{-1}iC_0) = C_0 f^{-1}iC_0$$

$$\Leftrightarrow \quad C_0 i C_0 C_0 f^{-1}iC_0 = C_0 f^{-1}iC_0$$

$$\Leftrightarrow \quad C_0 i C_0(C_0 f^{-1})iC_0 = (C_0 f^{-1})iC_0$$

$$\Leftrightarrow \quad C_0 i C_0(f^{-1}C_0)iC_0 = (f^{-1}C_0)iC_0, \text{ since } C_0 f^{-1} = f^{-1}C_0$$

$$\Leftrightarrow \quad (C_0 i C_0)f^{-1}(C_0 i C_0) = f^{-1}(C_0 i C_0)$$

$$\Leftrightarrow \quad Cf^{-1}C = f^{-1}C$$

5. Condition for semi-open set :

We show that condition for a set A to be semi-open is

$$\overline{A} = \overline{\text{int}(A)}$$

i.e. in our new relations

$$CA = CiA$$

Proof : We know that a subset A in a topological space X is semi-open if

$$A \subseteq CiA$$

Now CiA is a closed set containing A but smallest closed set containing A is CA.

Hence $CA \subseteq CiA$

But $iA \subseteq A \Rightarrow CiA \subseteq CA$

Thus $CA \subseteq CiA \subseteq CA$

Hence $CA = CiA$

Conversely, let $CA = CiA$

Thus $A \subseteq CA \subseteq CiA$

Hence A is semi-open.

Thus A is semi-open $\Leftrightarrow CA = CiA$

6. Condition for semi-continuity :

We show that condition for a function $f: X \rightarrow Y$; X, Y being topological spaces to be semi-continuous is

$$Cf^{-1}i = Cif^{-1}i$$

Proof : Function f is defined to be semi-continuous by N. Levine in [1], if for G open in Y, $f^{-1}(G)$ is a semi-open set in X.

Let A be any subset of Y. Then iA is open in Y. Hence $f^{-1}(iA)$ is a semi-open set in X. We have seen that for this required condition is

$$Cf^{-1}(iA) = Cif^{-1}(iA)$$

$$\Rightarrow \quad (Ci)(A) = (Cif^{-1}i)(A)$$

But A is any set, condition is

$$Cf^{-1}i = Cif^{-1}i$$

Consider the converse i.e. given above condition we need to show that f is semi continuous.

Let A be a set. Then

$$(Cf^{-1}i)(A) = (Cif^{-1}i)(A)$$

$$\Rightarrow \quad C(f^{-1}iA) = Ci(f^{-1}iA)$$

i.e. $f^{-1}(iA)$ is semi-open.

If A is an open set then $iA = A$, so $f^{-1}(iA) = f^{-1}(A)$ is semi-open. Thus inverse image of an open set w.r.t. f is semi-open i.e. f is semi-continuous.

7. Condition for semi-closed set :

We show that condition for A to be semi-closed is

$$iA = iCA$$

Proof : We know that a set A is semi-closed if

$$iCA \subseteq A$$

Thus iCA is an open set contained in A, but iA is the largest open set contained in A, hence $iCA \subseteq iA$.

But $A \subseteq CA \Rightarrow iA \subseteq iCA$

Hence $iA = iCA$

Conversely if $iA = iCA$, then

$$iCA = iA \subseteq A$$

i.e. A is semi-closed.

Thus set A is semi-closed $\Leftrightarrow iA = iCA$

8. A is semi-closed $\Leftrightarrow$ C_0A is semi-open.

Proof : First we observe that

$$CC_0 = (C_0 i C_0)C_0 = C_0 i$$

and $$C_0 iC = C_0 i(C_0 iC_0) = (C_0 i C_0)iC_0 = CiC_0$$

Now A is semi-closed if

$$iA = iCA$$

and C_0A is semi-open if

$$C(C_0A) = Ci(C_0A)$$

Hence

A is semi closed $\Leftrightarrow iA = iCA$

$$\Leftrightarrow \quad C_0(iA) = C_0(iCA)$$

$$\Leftrightarrow \quad (C_0 i)A = (C_0 iC)A$$

$$\Leftrightarrow \quad CC_0A = (CiC_0)A$$

$$\Leftrightarrow \quad C(C_0A) = Ci(C_0A)$$

$\Leftrightarrow$ C_0A is semi-open.

9. α-closed set :

As defined by Njastad in [3], a set A is α-open if $A \subseteq iCi(A)$

A is α-closed if

$$CiC(A) \subseteq A$$

Now, we show that a nowhere dense set is α-closed.

Proof : Let a set A be nowhere dense i.e.

$$\text{int}\,\bar{A} = \phi$$

i.e. $iCA = \phi$

Now $iCA = \phi \subseteq A$. So A is semi closed.

Also $CiCA = C(\phi) = \phi \subseteq A$

Hence A is α-closed.

Conclusion :

In this paper, we have introduced some notations in study of topological spaces which are quite useful. With its help conditions of continuity, semi-continuity, etc. have been put in new elegant forms. Conditions of semi-open, semi-closed sets have been also derived.

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