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Optimization through Quick Simplex Method

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Abstract:

This research paper proposes the Quick simplex method to solve the Quadratic Programming Problem (QPP) for the very first time. The same had been introduced to solve the Linear Programming Problem (LPP) by Vaidya et al. In this paper, we used the Quick simplex method to solve linearly factorized quadratic optimization problem. These types of problems are concerned with the non-linear programming problems of optimizing (maximizing/minimizing) the objective function subject to a set of linear inequality constraints. The proposed method is more efficient as compared to traditional simplex method as it attempts to replace more than one basic variable simultaneously.

Keywords: Quick Simplex Method, Quadratic Optimization, Optimal Solution, Key determinant.

Introduction:

Quadratic Programming Problem (QPP) is one of the simplest forms of Non-Linear Programming and used to solve the problems of optimizing an objective function of quadratic form or a linearly factorized quadratic objective function. The objective function of QPP can contain bilinear terms and the constraints are linear and can be both equations and inequalities. QPP is used widely in Signal and image processing, to perform the Least - squares approximations and estimations, to optimize financial portfolios and to control scheduling, etc. Wolfe[1] developed simplex method to solve QPP. To deal with these types of problems, several methods have been proposed. Some of them are mentioned here which suits to our research. Jayalakshmi, M. et al[2] introduced a method to solve quadratic programming problems having linearly objective function. Further, Jain et al[3,4] developed techniques for modeling of QPP by Gauss elimination technique and nature inspired modified Fourier elimination technique. Game problems was solved through Quick Simplex Algorithm by Vaidya et al[5]. Vaidya et al[6] presented an optimal solution of LPP by Quick Simplex Algorithm. A comparison between various entering vector criteria with quick simplex algorithm for optimal solution to the linear programming problem was discussed by Vaidya et al[7]. Graphical view of quick simplex method was described by Vaidya et al[8]. Vaidya et al[9] discussed the application of quick simplex method on two phase method. Vaidya[10] presented application of quick simplex method on the dual simplex method. Also, Vaidya et al[11] explained graphical view of quick simplex method to solve LPP. In this research paper, we are proposing Quick simplex method for QPP for the first time in the literature.

Preliminaries:

Linear Factorized Quadratic Optimization Problem:

Consider a quadratic optimization problem in which the objective function can be expressed as the multiplication of two linear functions as follows:

$$\text{Maximize } f(x) = \frac{1}{2} x^T Q x + c^T x + d$$
$$= (c_1^T x + \alpha)(c_2^T x + \beta)$$

such that $Ax \leq b$

and $x \geq 0$

Where d, α, β are scalars, A is an $m \times n$ matrix and $x^T, c, c_1, c_2 \in R^n$, $b^T \in R^m$; Q be a symmetric matrix of order $n \times n$.

Quick Simplex Algorithm [10]:

The steps of quick simplex algorithm to solve QPP are as under:

1. We have to form as many separate linear programming problems as given in the linearly factorized optimization problem under consideration.

2. The objective function of every linear program must be in maximization form. If it is not so, then we have to convert it into maximize form by using the standard result i.e.,

$$\text{Maximize } Z_i = -\text{ Minimize } Z_i$$

3. The components of requirement vector i.e., b_i must be positive always. If any one of the b_i is not positive, then we have to make it positive by multiplying the corresponding inequality by -1.

4. We have to convert all the inequalities involved in the given problem into strict equations by using slack, surplus or artificial variables (if required).

5. Prepare an initial simplex table as usual traditional method and obtain an initial basic feasible solution.

6. Calculate the components of net evaluation row i.e., $Z_j - C_j$ ($j = 1, 2, 3, \ldots.., n$) as usual.

7. We have to select most negative element in net evaluation row. Now, we find ratio μ_{rj} (say) for positive a_{rj}. If all the entries of the j^{th} column are negative then go for next most negative net evaluation. If only one net evaluation is negative and all the entries in the j^{th} column are negative, then the problem has an unbounded solution and procedure ends here otherwise go to the next step.

8. Select the minimum ratio among μ_{rj}. Let it be μ_{kj}. If there is a tie in the ratios, one can break the tie arbitrarily.

9. Pivotal element will be a_{kj} corresponding to μ_{kj}.

10. If number of pivotal elements is less than m, then we have to check if any more negative net evaluation is left. If it is so then go to point 8, ignoring row and column of pivotal element otherwise go to the next step.

11. The incoming variable is the variable corresponding to the column of pivotal element. Incoming variables replace the variables corresponding to the row of the pivotal elements. It is to be noted that the number of pivotal elements say r. Then we define key determinant of order $r \times r$ by using sub matrix of matrix A whose diagonal elements are the pivotal elements.

12. Calculate the entries for new basis.

13. If all the entries are positive, then the above replacement of such variables is permitted and one can proceed further. If not so, then we have to follow point no. 15.

14. Now, we have to develop a new simplex table by replacing the r variables simultaneously.

15. We have to check for optimality of the solution by using net evaluations.

    (a) If all the entries of net evaluations are positive, then this is an optimal one.

    (b) If not so, then go to point no. 6.

Illustration:

Consider the linearly factorized quadratic programming problem

$$\begin{aligned} &\text{Max. } Z = (5x + 3y)(5x + 2y + 1) \\ &\text{s.t.} \quad 3x + 5y \leq 15 \\ &\qquad\quad 5x + 2y \leq 10 \\ &\text{and} \quad x, y \geq 0 \end{aligned}$$

At first, we have to form two linear programming problems from the above quadratic programming problem.

Problem I:

$$\begin{aligned} &\text{Max. } Z_1 = (5x + 3y) \\ &\text{s.t.} \quad 3x + 5y \leq 15 \\ &\qquad\quad 5x + 2y \leq 10 \\ &\text{and} \quad x, y \geq 0 \end{aligned}$$

Problem II:

$$\begin{aligned} &\text{Max. } Z_2 = (5x + 2y + 1) \\ &\text{s.t.} \quad 3x + 5y \leq 15 \\ &\qquad\quad 5x + 2y \leq 10 \\ &\text{and} \quad x, y \geq 0 \end{aligned}$$

After adding slack variables S_1 and S_2, above problems become

Problem I:

$$\begin{aligned} &\text{Max. } Z_1 = 5x + 3y + 0S_1 + 0S_2 \\ &\text{s.t.} \quad 3x + 5y + S_1 + 0S_2 = 15 \\ &\qquad\quad 5x + 2y + 0S_1 + S_2 = 10 \\ &\text{and} \quad x, y, S_1, S_2 \geq 0 \end{aligned}$$

Problem II:

$$\begin{aligned} &\text{Max. } Z_2 = 5x + 2y + 1 + 0S_1 + 0S_2 \\ &\text{s.t.} \quad 3x + 5y + S_1 + 0S_2 = 15 \\ &\qquad\quad 5x + 2y + 0S_1 + S_2 = 10 \\ &\text{and} \quad x, y, S_1, S_2 \geq 0 \end{aligned}$$

Solution of problem I by Quick Simplex Algorithm

		C_j	5	3	0	0	R_1	R_2
C_B	X_B	b	x	y	S_1	S_2		
0	S_1	15	3	5	1	0	5	3
0	S_2	10	5	2	0	1	2	5
$Z_j - C_j \rightarrow$			-5	-3	0	0		

It is very much obvious from the net evaluations that we can introduce two variables first x then y as incoming variables and they can be replaced by S_2 and S_1. In Quick Simplex algorithm, one can directly get third simplex table. Introduction of entering variables should be in such a manner so that pivotal elements may occur in different rows.

We can directly obtain third simplex table by using the determinant $R = \begin{vmatrix} 5 & 3 \\ 2 & 5 \end{vmatrix} = 19$.

(We have to adjust pivotal elements in diagonal form if not so)

Here R is a key determinant and $|R|$ is used as a denominator to calculate each element in the third simplex table. Now, we can calculate the values of the basic variables in the third simplex table as follows:

$$x = \frac{\begin{vmatrix} 5 & 15 \\ 2 & 10 \end{vmatrix}}{R} = \frac{20}{19}, \quad y = \frac{\begin{vmatrix} 15 & 3 \\ 10 & 5 \end{vmatrix}}{R} = \frac{45}{19}$$

Since the values of the basic variables x and y are positive and non-negative constraints are satisfied, therefore such transformation is permitted.

Now, new S_1 column for the third simplex table will be as follows:

$$x = \frac{\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix}}{R} = \frac{5}{19}, \quad y = \frac{\begin{vmatrix} 5 & 1 \\ 2 & 0 \end{vmatrix}}{R} = -\frac{2}{19}$$

Also, new S_2 column for the third simplex table will be as follows:

$$x = \frac{\begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix}}{R} = -\frac{3}{19}, \quad y = \frac{\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix}}{R} = \frac{5}{19}$$

Final Table

		C_j	5	3	0	0
C_B	X_B	b	x	y	S_1	S_2
3	y	$\frac{45}{19}$	0	1	$\frac{5}{19}$	$-\frac{3}{19}$
5	x	$\frac{20}{19}$	1	0	$-\frac{2}{19}$	$\frac{5}{19}$
$Z_j - C_j \rightarrow$			0	0	$\frac{5}{19}$	$\frac{16}{19}$

Since all the entries of net evaluations are positive, therefore this is an optimal solution. An optimal solution of the problem I is as follows:

$$x = \frac{20}{19}, y = \frac{45}{19} \text{ and Max. } Z_1 = \frac{235}{19}$$

After adding slack variables S_1 and S_2, problem II reduces to

$$\begin{aligned} &\text{Max. } Z = 5x + 2y + 1 + 0S_1 + 0S_2 \\ &\text{s.t.} \quad 3x + 5y + S_1 + 0S_2 = 15 \\ &\qquad\;\; 5x + 2y + 0S_1 + S_2 = 10 \\ &\text{and} \quad x, y, S_1, S_2 \geq 0 \end{aligned}$$

Solution of problem II by Quick Simplex Algorithm

		C_j	5	2	0	0	R_1	R_2
C_B	X_B	b	x	y	S_1	S_2		
0	S_1	15	3	5	1	0	5	3
0	S_2	10	5	2	0	1	2	5
$Z_j - C_j \rightarrow$			-5	-2	0	0		

It is very much obvious from the net evaluations that we can introduce two variables first x then y as incoming variables and they can be replaced by S_2 and S_1. In Quick Simplex algorithm, one can directly get third simplex table. Introduction of entering variables should be in such a manner so that pivotal elements may occur in different rows.

We can directly obtain third simplex table by using the determinant $R = \begin{vmatrix} 5 & 3 \\ 2 & 5 \end{vmatrix} = 19.$

(We have to adjust pivotal elements in diagonal form if not so)

Here R is a key determinant and $|R|$ is used as a denominator to calculate each element in the third simplex table. Now, we can calculate the values of the basic variables in the third simplex table as follows:

$$x = \frac{\begin{vmatrix} 5 & 15 \\ 2 & 10 \end{vmatrix}}{R} = \frac{20}{19},\quad y = \frac{\begin{vmatrix} 15 & 3 \\ 10 & 5 \end{vmatrix}}{R} = \frac{45}{19}$$

Since the values of the basic variables x and y are positive and non-negative constraints are satisfied, therefore such transformation is permitted.

Now, new S_1 column for the third simplex table will be as follows:

$$x = \frac{\begin{vmatrix} 1 & 3 \\ 0 & 5 \end{vmatrix}}{R} = \frac{5}{19},\quad y = \frac{\begin{vmatrix} 5 & 1 \\ 2 & 0 \end{vmatrix}}{R} = -\frac{2}{19}$$

Also, new S_2 column for the third simplex table will be as follows:

$$x = \frac{\begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix}}{R} = -\frac{3}{19},\quad y = \frac{\begin{vmatrix} 5 & 0 \\ 2 & 1 \end{vmatrix}}{R} = \frac{5}{19}$$

Final Table

		C_j	5	2	0	0
C_B	X_B	b	x	y	S_1	S_2
2	y	$\frac{45}{19}$	0	1	$\frac{5}{19}$	$-\frac{3}{19}$
5	x	$\frac{20}{19}$	1	0	$-\frac{2}{19}$	$\frac{5}{19}$
$Z_j - C_j \rightarrow$			0	0	0	1

Since all the entries of net evaluations are positive, therefore this is an optimal solution. An optimal solution of the problem II is as follows:

$$x = \frac{20}{19}, y = \frac{45}{19} \text{ and Max. } Z_2 = \frac{209}{19}$$

Therefore, an optimal solution of linearly factorized quadratic programming problem is as follows:

$$x = \frac{20}{19}, y = \frac{45}{19} \text{ and Max. } Z = Z_1 Z_2 = \left(\frac{235}{19}\right)\left(\frac{209}{19}\right) = \frac{49115}{361}$$

Conclusion:

One can easily verify the above solution by traditional simplex method, graphical method, Gauss elimination technique, Modified Fourier elimination technique, etc. in case of linear programming problems to validate our claim. Quick simplex method involves either lesser number of iterations or at the most equal number of iterations as compared to traditional simplex method because Quick Simplex method replaces more than one basic variable simultaneously.

Conflicts of interest:

The authors declares no conflicts of interest in publishing this research paper.

References:

1. Wolfe, P. : "The simplex method for quadratic programming", Econometrica; 27(3), 382-39, (1959).

2. Jayalakshmi, M. and Pandian, P. : "A method for solving quadratic programming problems having linearly factorized objective function", International Journal of Modern Engineering Research 4(8), 20-24, (2019).

3. Jain, S., Mangal, A. and Sangeeta : "Modeling of Gauss Elimination Technique for Quadratic Optimization", Advances in Modeling and Analysis A.,31-34, (2019).
https://doi.org/10.18280/ama_a.562-401

4. Jain, S., Mangal, A. and Sangeeta : "Modeling of nature inspired modified Fourier elimination technique for quadratic optimisation", International Journal of Swarm Intelligence, Vol. 5, No. 2, 182-190 (2020).
https://doi.org/10.1504/IJSI.2020.111162

5. Vaidya, N.V. and Kasturiwale, N.N. : "Solving Game Problems Using a Quick Simplex Algorithm a New Method", Int. J Latest Trend Math, Vol. 4, No. 1, 165-182, (2014).

6. Vaidya, N.V. and Kasturiwale, N.N. : "Quick Simplex Algorithm for Optimal Solution to the Linear Programming Problem along with Theoretical Proof of Formulae", Int. J Latest Trend Math, Vol. 4, No. 2, 183-200, (2014).

7. Vaidya, N.V. and Kasturiwale, N.N. : "Comparison between various entering vector criteria with quick simplex algorithm for optimal solution to the linear programming problem", International Journal of Mathematics and Statistics Invention, Volume 2, Issue 6, 17-36, (2014).

8. Vaidya, N.V. and Kasturiwale, N.N. : "Graphical View of Quick Simplex Method (A New Approach) to Solve L.P.P.", Proceedings of 2nd International Conference On Recent Trends & Innovation in Engineering & Technology, 19th - 20th December 2014, 1-7, (2014).

9. Vaidya, N.V. and Kasturiwale, N.N. : "Application of Quick Simplex Method (A New Approach) On Two Phase Method", British Journal of Mathematics & Computer Science, 16(1), 1-15, (2016).

10. Vaidya, N.V. : "Application of Quick Simplex Method on the Dual Simplex Method (A New Approach)", Journal of Advances in Mathematics and Computer Science, 24(5), 1-9, (2017).

11. Vaidya, N.V., Pidurkar, S.R., Shanti, M. and Uparkar, S.S. : "Graphical View of Quick Simplex Method to Solve Linear Programming Problem", International Journal of Advanced Science and Technology, Vol. 29, No. 6, 5694-5710, (2020).