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Almost H^*-Normal Spaces

by **Neeraj Kumar Tomar[1], Fahed Zulfeqarr[2] & Amit Ujlayan[3]**
Department of Mathematics,
Gautam Buddha University, Greater Noida - 201312, India
Emails: [1] neer8393@gmail.com, [2] fahed@gbu.ac.in, [3] amitujlayan@gbu.ac.in

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Abstract:

The aim of this paper is to introduced and study a new class of almost normal spaces, called almost H^*-normal spaces by using H^*-open sets due to Nidhi Sharma et al. obtained several properties of such a space. Moreover, we obtain some new characterizations and preservation theorems of almost H^*-normal spaces.

Keywords: H^*-open; H^*-closed; M-H^*-closed; M-H^*-open; almost H^*-irresolute functions; almost H^*-normal spaces.

2020 Mathematics Subject Classification: 54A05, 54C08, 54C10, 54D15.

Introduction:

In this paper, we introduced the concept of almost H^*-normal by using H^*-open set due to Nidhi Sharma et al. [1] and obtained several properties of such a space. In 1970, Singal and Arya [2] introduced the concept of almost normal spaces as a generalization of normal-spaces by using regularly closed sets and obtained several properties of such a space. In 2016, Hamant Kumar and M.C. Sharma [3] introduced a new class of spaces, namely almost γ-normal and mildly γ-normal spaces are weaker form of γ-normal spaces. We show that these normal spaces, namely almost γ-normal and mildly γ-normal spaces are regularly open hereditary are give relationship of almost γ-normal and mildly γ-normal spaces and some known weaker form of almost normal and mildly normal spaces and obtain characterizations and preservation theorems of almost γ-normal and mildly γ-normal spaces. We introduced the concepts of gH^*-closed, rgH^*-closed, regularly H^*-closed sets, M-H^*-closed, M-H^*-open, almost H^*-irresolute functions. Moreover, we obtain some new characterizations and preservation theorems of almost H^*-normal spaces. Throughout this paper, (X, τ), (Y, σ) spaces always mean topological spaces X, Y respectively on which no separation axioms are assumed unless explicitly stated.

1. Preliminaries and Notations:

In what follows, spaces always mean topological spaces on which no separation axioms are assumed unless explicitly stated and $f : (X, \tau) \rightarrow (Y, \sigma)$ (or simply $f : X \rightarrow Y$) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by cl(A) and int(A), respectively.

We now define some basic notions which will be used throughout. For a good understanding of them, readers are referred to see **[4, 5, 6, 7, 8, 9],** respectively.

1.1 Definition: A subset A of a topological space X is said to be

(1) **regular closed** [4] if $A = \text{cl}(\text{int}(A))$.

(2) **semi closed** [5] if $\text{int}(\text{cl}(A)) \subset A$.

(3) ***w*-closed** [6] if $\text{cl}(A) \subseteq U$, whenever $A \subseteq U$ and U is an semi-open in X.

(4) **α-closed set** [7] if $\text{cl}(\text{int}(\text{cl}(A))) \subseteq A$.

(5) **α^*-set** [8] if $\text{int}(\text{cl}(\text{int}(A))) = \text{int}(A)$.

(6) ***C*-set** [8] if $A = U \cap V$, where U is an open set and V is an α^*-set in X.

(7) ***h*-closed** [9] if s-cl(A) $\subseteq U$, whenever $A \subseteq U$ and U is a w-open in X.

(8) ***gh*-closed** if h-cl(A) $\subseteq U$, whenever $A \subseteq U$ and U is a h-open in X.

(9) **regular-*h*-open** if there is a regular open set U such that $U \subseteq A \subseteq h$-cl($U$).

(10) ***rgh*-closed** if h-cl(A) $\subseteq U$, whenever $A \subseteq U$ and U is regularly h-open in X.

(11) ***hCg*-closed** if h-cl(A) $\subseteq U$, whenever $A \subseteq U$ and U is a C-set in X.

The complement of a regular-closed (resp. semi-closed, w-closed, α-closed, h-closed, gh-closed, rgh-closed, and hCg-closed) set is called **regular-open** (resp. **semi-open, *w*-open, α-open, *h*-open, *gh*-open, *rgh*-open, and *hCg*-open**).

The complement of regular-h-open set is called **regular-*h*-closed** set.

The intersection of all h-closed (resp. semi-closed) sets containing A is called the ***h*-closure** (resp. **semi-closure**) of ***A*** and is denoted by ***h*-cl(*A*)** (resp. ***s*-cl(*A*)**). The ***h*-interior** (resp. **semi-interior**) of ***A***, denoted by ***h*-int(*A*)** (resp. ***s*-int(*A*)**) is defined to be the union of all h-open (resp. semi-open) sets contained in A.

1.2 Definition: A subset A of a topological space X is said to be ***H****-closed** if h-cl(A) $\subset U$ whenever $A \subset U$ and U is hCg-open in X. The complement of H^*-closed set is said to be ***H****-open**.

The intersection of all H^*-closed sets containing A, is called the ***H****-closure of *A*** and is denoted by ***H****-cl(*A*)**. The ***H****-interior of *A***, denoted by ***H****-int(*A*)** is defined to be the union of all H^*-open sets contained in A.

The family of all H^*-open (resp. H^*-closed, regular open, regular closed, semi-open, semi-closed) sets of a space X is denoted by $\boldsymbol{H^*O(X)}$ (resp. $\boldsymbol{H^*C(X)}$, $\boldsymbol{RO(X)}$, $\boldsymbol{RC(X)}$, $\boldsymbol{SO(X)}$, $\boldsymbol{SC(X)}$).

1.3 Definition: A subset A of a topological space X is said to be

(1) ***g*-closed** [10] if cl(A) $\subset U$ whenever $A \subset U$ and $U \in \tau$.

(2) **generalized *H****-closed** (briefly ***gH****-closed**) if H^*-cl(A) $\subset U$ whenever $A \subset U$ and U is H^*-open.

(3) ***H**** **generalized-closed** (briefly ***H*g*-closed**) if H^*-cl(A) $\subset U$ whenever $A \subset U$ and U is open in X.

(4) **regular-H^*-open** if there is a regular open set U such that $U \subseteq A \subseteq H^*\text{-cl}(U)$.
The complement of g-closed (resp. gH^*-closed, H^*g-closed) set is said to be **g-open** (resp. **gH^*-open, H^*g-open**).

The complement of regular-H^*-open is said to be **regular-H^*-closed**.

1.4 Remark: We have the following implications for the properties of subsets:

$$\begin{array}{l} \text{closed} \Rightarrow H^*\text{-closed} \Leftrightarrow H^*g\text{-closed} \Leftrightarrow gH^*\text{-closed} \\ \quad\downarrow \\ g\text{-closed} \end{array}$$

However the converse of the above are not true may be seen by the following example.

1.5 Remark: We have the following implications for the properties of subsets:

$$\begin{array}{ccccccc} \text{closed} & \rightarrow & \alpha\text{-closed} & \rightarrow & g\alpha\text{-closed} & \rightarrow & rg\alpha\text{-closed} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & H^*\text{-closed} & & gH^*\text{-closed} & & rgH^*\text{-closed} \end{array}$$

1.6 Example: Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then $A = \{c\}$ is h-closed set as well as H^*-closed set but not closed set in X.

1.7 Example: Let $X = \{a, b, c, d, e\}$ and $\tau = \{\phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}, X\}$. Then $A = \{a, d, e\}$ is rgh-closed set as well as rgH^*-closed set but not gh-closed set and not gH^*-closed set in X.

1.8 Example: Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then $A = \{c\}$ is gH^*-closed set but not closed set in X.

1.9 Example: Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$. Then

(1) closed sets in X are $\phi, X, \{b, c, d\}, \{a, c, d\}, \{c, d\}, \{d\}$.

(2) g-closed sets in X are $\phi, X, \{d\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$.

(3) H^*-closed sets in X are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

(4) gH^*-closed sets in X are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

(5) H^*g-closed sets in X are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{b, c, d\}$.

1.10 Lemma: Let A be a subset of a space X and $x \in X$. The following properties hold for H^*-cl(A):

(i) $x \in H^*$-cl(A) if and only if $A \cap U \neq \phi$ for every $U \in H^*O(X)$ containing x.

(ii) A is H^*-closed if and only if $A = H^*$-cl(A).

(iii) H^*-c1(A) $\subset H^*$-c1(B) if $A \subset B$.

(iv) H^*-c1(H^*-c1(A)) = H^*-c1(A).

(v) H^*-c1(A) is H^*-closed.

1.11 Lemma: A subset A of a topological space X is gH^*-open in X if and only if $F \subset H^*$-int(A) whenever $F \subset A$ and F is closed in X.

2. Almost H^*-normal spaces:

2.1 Definition: A topological space X is said to be almost normal [2] (resp. **almost *H**-normal**) if for every pair of disjoint closed sets A and B, one of which is closed and other is regularly closed, there exist disjoint open (resp. ***H**-open**) sets U and V of X such that $A \subset U$ and $B \subset V$.

2.2 Example: Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$. Then $A = \{b\}$ is closed and $B = \{a\}$ is regularly closed sets there exist disjoint open sets $U = \{b, c, d\}$ and $V = \{a\}$ of X such that $A \subset U$ and $B \subset V$. Hence X is almost normal as well as almost H^*-normal because every open set is H^*-open set.

By the definitions and examples stated above, we have the following diagram:

normal $\Rightarrow$ almost-normal $\Rightarrow$ almost H^*-normal

2.3 Lemma: A subset A of a topological space X is rgH^*-open if and only if $F \subset H^*$-int(A) whenever F is regularly closed and $F \subset A$.

2.4 Theorem: For a topological space X the following are equivalent:

(a) X is almost H^*-normal.

(b) For every closed sets A and every regularly closed set B there exist disjoint gH^*-open sets U and V such that $A \subset U$ and $B \subset V$.

(c) For every closed sets A, and every regularly closed set B there exist disjoint rgH^*-open sets U and V such that $A \subset U$ and $B \subset V$.

(d) For every closed set A and every regularly open set B containing A, there exists a gH^*-open set U such that $A \subset U \subset H^*$-cl(U) $\subset B$.

(e) For every closed set A and every regularly open set B containing A, there exists a rgH^*-open set U such that $A \subset U \subset H^*$-cl(U) $\subset B$.

(f) For every pair of disjoint sets A and B one of which closed and other is regularly closed, there exist H^*-open sets U and V such that $A \subset U$ and $B \subset V$ and $U \cap V = \phi$.

Proof: (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (d) $\Rightarrow$ (e), (c) $\Rightarrow$ (d), (e) $\Rightarrow$ (f), (f) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Let X be a almost H^*-normal. Let A be a closed and B be a regularly closed sets in X. By assumption, there exist disjoint H^*-open sets U and V such that $A \subset U$ and $B \subset V$. Since every H^*-open set is gH^*-open set. U, V are gH^*-open sets such that $A \subset U$ and $B \subset V$.

(b) $\Rightarrow$ (c): Let A be a closed and B be a regularly closed sets in X. By assumption, there exist disjoint gH^*-open sets U and V such that $A \subset U$ and $B \subset V$. Since every gH^*-open set is rgH^*-open set, U and V are rgH^*-open sets such that $A \subset U$ and $B \subset V$.

(d) $\Rightarrow$ (e): Let A be any closed set and B be any regularly open set containing A. By assumption, there exists a gH^*-open set U of X such that $A \subset U \subset H^*$-cl(U) $\subset B$. Since every gH^*-open set is rgH^*-open set, there exists a rgH^*-open set U of X such that $A \subset U \subset H^*$-cl(U) $\subset B$.

(c) $\Rightarrow$ (d): Let A be any closed set and B be a regularly open set containing A. By assumption, there exist disjoint rgH^*-open sets U and V such that $A \subset U$ and $X - B \subset V$. By **Lemma 2.3**, we get

$X - B \subset H^*\text{-int}(V)$ and $H^*\text{-cl}(U) \cap H^*\text{-int}(V) = \phi$. Hence,

$A \subset U \subset H^*\text{-cl}(U) \subset X - H^*\text{-int}(V) \subset B$.

(e) $\Rightarrow$ (f): For any closed set A and any regularly open set B containing A. Then $A \subset X - B$ and $X - B$ is a regularly closed. By assumption, there exists a rgH^*-open set G of X such that, $A \subset G \subset H^*\text{-cl}(G) \subset X - B$. Put $U = H^*\text{-int}(G)$, $V = X - H^*\text{-cl}(G)$. Then U and V are disjoint H^*-open sets of X such that $A \subset U$ and $B \subset V$.

(f) $\Rightarrow$ (a): is obvious.

2.5 Definition: A function $f: X \rightarrow Y$ is called **rc-continuous**[10] if for each regular closed set F in Y, $f^{-1}(F)$ is regularly closed in X.

2.6 Definition: A function $f: X \rightarrow Y$ is called ***M-H*-open*** (resp. ***M-H*-closed)*** if $f(U) \in H^*O(Y)$ (resp. $f(U) \in H^*C(Y)$) for each $U \in H^*O(X)$ (resp. $U \in H^*C(X)$).

2.7 Definition: A function $f: X \rightarrow Y$ is called **almost *H**-irresolute** if for each $x \in X$ and each H^*-neighbourhood V of $f(X)$, $H^*\text{-cl}(f^{-1}(V))$ is a H^*-neighbourhood of x.

3. Preservation theorems:

3.1 Theorem: A function $f: X \rightarrow Y$ is continuous M-H^*-open rc-continuous and almost H^*-irresolute surjection from an almost H^*-normal space X, then Y is almost H^*-normal.

Proof: Let A be a closed set and B be a regularly open set containing A. Then by rc-continuity of f, $f^{-1}(A)$ is a closed set contained in the regularly open set $f^{-1}(B)$. Since X is almost H^*-normal, there exists a H^*-open set V in X such that $f^{-1}(A) \subset V \subset H^*\text{-cl}(V) \subset f^{-1}(B)$ by **Theorem 2.4,** Then, $f(f^{-1}(A)) \subset f(V) \subset f(H^*\text{-cl}(V)) \subset f(f^{-1}(B))$. Since f is M-H^*-open and almost H^*-irresolute surjection, it follows that $f(V) \in H^*O(Y)$, we obtain $A \subset f(V) \subset H^*\text{-cl}(f(V)) \subset B$. Then again by **Theorem 2.4,** Y is almost H^*- normal.

3.2 Theorem: If $f : X \to Y$ is rc-continuous M-H^*-closed map from an almost H^*-normal space X onto a space Y, then Y is almost H^*-normal.

Proof: Easy to verify.

Conclusion:

In this paper, we introduce and study a new class of spaces, namely almost H^*-normal spaces by using H^*-open sets. The relationship among normality, almost H^*-normality which is weaker form of normality. Also we obtained some characterization of almost H^*-normal spaces, properties of the forms of M-H^*-closed functions. Of course, the entire content will be a successful tool for the researchers for finding the way to obtain the results in the context of such types of normal spaces. This idea can be extended to topological ordered, bitopological ordered and fuzzy topological spaces etc.

Conflicts of Interest:

We certify that this work is original, has never been published before, and is not being considered for publication anywhere else at this time. This publication is free from any conflicts of interest. As the Corresponding Author[1], I certify that each of the listed authors has read the paper and given their approval for submission.

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