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Definite Integrals involving Modified Bessel Functions

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Abstract:

In this paper, we develop fifteen definite integral involving Modified Bessel function of first kind. Furthermore, we also develop the generalized formula for the same using Inspection method.

Keywords: Bessel function, hypergeometric function, Pochhammer symbol

2020 Mathematics Subject Classifications: 33B50, 33C05, 33C10, 33D50, 33D60, 33D67.

1. Introduction:

Brychkov (see[2], p.199(4.7.4.1)) has derived the following formula;

$$\int_0^1 z \log z \, J_0(az)dz = -\frac{1}{a^2}[J_0(a) - 1] \tag{1.1}$$

First kind Bessel function is denoted by $J_\eta(y)$, and it is defined as;

$$J_\upsilon(y) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\upsilon+1)} \left(\frac{y}{2}\right)^{2m+\upsilon} \tag{1.2}$$

where Γ is the gamma function.

The first kind of modified Bessel function is defined as

$$I_\sigma(z) = l^{-\sigma} J_\sigma(lz) = \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\sigma+1)} \left(\frac{z}{2}\right)^{2m+\sigma} \tag{1.3}$$

Generalized hypergeometric function ${}_pF_q(a_1,..., a_p ; b_1,..., b_q ; z)$ is a function which can be described in the form of a hypergeometric series, i.e., a series for which the ratio of consecutive terms can be written

$$\frac{c_{m+1}}{c_m} = \frac{P(m)}{Q(m)} = \frac{(m+a_1)(m+a_2)\ldots(m+a_p)}{(m+b_1)(m+b_2)\ldots(m+b_q)(m+1)} z \tag{1.4}$$

Where $m + 1$ in the denominator is present for documentary causes of notation given by Koepf (see[9], p.12(2.9)), and the developing generalized hypergeometric function is written as

$${}_pF_q\left[\begin{matrix} a_1, a_2, \ldots, a_p \;; \\ b_1, b_2, \ldots, b_q \;; \end{matrix} \; z\right] = \sum_{m=0}^{\infty} \frac{(a_1)_m (a_2)_m \ldots (a_p)_m z^m}{(b_1)_m (b_2)_m \ldots (b_q)_m \, m!} \tag{1.5}$$

where the parameters $b_1, b_2,..., b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $|z| < 1$ if $p = q + 1$, diverges for all $z, z \neq 0$ if $p > q + 1$, given by Luke (see [10], p.156(3)).

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function (see [8], p. 123-162).

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial, falling sequential product, or lower factorial) is defined as;

$$(z)_n = z(z-1)(z-2)......(z-n+1) = \prod_{k=1}^{n} (z-k+1) = \prod_{k=0}^{n-1} (z-k) \tag{1.6}$$

2. Main Formulas:

In this section, we establish fifteen definite integral involving modified Bessel function, as follows:

$$\int_0^1 z \log z \, I_0(az) \, dz = \frac{1 - I_0(a)}{a^2} \tag{2.1}$$

$$\int_0^1 z^2 \log z \, I_0(az) \, dz = -\frac{1}{9} \, {}_2F_3\left(\frac{3}{2}, \frac{3}{2}; 1, \frac{5}{2}, \frac{5}{2}; \frac{a^2}{4}\right) \tag{2.2}$$

$$\int_0^1 z^3 \log z \, I_0(az) \, dz = \frac{-(a^2+4)I_0(a) + 4a\, I_1(a) + 4}{a^4} \tag{2.3}$$

$$\int_0^1 z^4 \log z \, I_0(az) \, dz = -\frac{1}{25} \, {}_2F_3\left(\frac{5}{2}, \frac{5}{2}; 1, \frac{7}{2}, \frac{7}{2}; \frac{a^2}{4}\right) \tag{2.4}$$

$$\int_0^1 z^5 \log z \, I_0(az) \, dz = \frac{a^4(-I_4(a)) - 8(a^2+8)\, I_0(a) + 48a\, I_1(a) + 64}{a^6} \tag{2.5}$$

$$\int_0^1 z^6 \log z \, I_0(az) \, dz = -\frac{1}{49} \, {}_2F_3\left(\frac{7}{2}, \frac{7}{2}; 1, \frac{9}{2}, \frac{9}{2}; \frac{a^2}{4}\right) \tag{2.6}$$

$$\int_0^1 z^7 \log z\, I_0(az)\, dz$$

$$= \frac{-(a^2+24)(a^4+60a^2+96)\, I_0(a) + 12a(a^4+40a^2+352)\, I_1(a) + 2304}{a^8} \tag{2.7}$$

$$\int_0^1 z^{11} \log z\, I_0(az)\, dz$$

$$= \frac{1}{a^{12}} [20\{a(a^8+144a^6+9024a^4+236544a^2+1683456)\ I_1(a)+737280\}$$

$$- (a^{10}+260a^8+23680a^6+952320a^4+13148160a^2+14745600) I_0(a)] \tag{2.8}$$

$$\int_0^1 z^{14} \log z\, I_0(az)\, dz = -\frac{1}{225}\, {}_2F_3\left(\frac{15}{2}, \frac{15}{2}; 1, \frac{17}{2}, \frac{17}{2}; \frac{a^2}{4}\right) \tag{2.9}$$

$$\int_0^1 z^{20} \log z\, I_0(az)\, dz = -\frac{1}{441}\, {}_2F_3\left(\frac{21}{2}, \frac{21}{2}; 1, \frac{23}{2}, \frac{23}{2}; \frac{a^2}{4}\right) \tag{2.10}$$

$$\int_0^1 z^{30} \log z\, I_0(az)\, dz = -\frac{1}{961}\, {}_2F_3\left(\frac{31}{2}, \frac{31}{2}; 1, \frac{33}{2}, \frac{33}{2}; \frac{a^2}{4}\right) \tag{2.11}$$

$$\int_0^1 z^{40} \log z\, I_0(az)\, dz = -\frac{1}{1681}\, {}_2F_3\left(\frac{41}{2}, \frac{41}{2}; 1, \frac{43}{2}, \frac{43}{2}; \frac{a^2}{4}\right) \tag{2.12}$$

$$\int_0^1 z^{42} \log z\, I_0(az)\, dz = -\frac{1}{1849}\, {}_2F_3\left(\frac{43}{2}, \frac{43}{2}; 1, \frac{45}{2}, \frac{45}{2}; \frac{a^2}{4}\right) \tag{2.13}$$

$$\int_0^1 z^{50} \log z\, I_0(az)\, dz = -\frac{1}{2601}\, {}_2F_3\left(\frac{51}{2}, \frac{51}{2}; 1, \frac{53}{2}, \frac{53}{2}; \frac{a^2}{4}\right) \tag{2.14}$$

and

$$\int_0^1 z^{55} \log z\, I_0(az)\, dz = -\frac{1}{3136}\, {}_2F_3\left(28, 28; 1, 29, 29; \frac{a^2}{4}\right) \tag{2.15}$$

Proof : We first prove our assertion (2.1), we make use the properties of Modified Bessel function of first kind as below;

$$\int_0^1 z \log z \, I_0(az)\, dz = \left[\frac{1}{2} z^2 \log z \; {}_0\tilde{F}_1 \, ; \left(2 \, ; \frac{a^2 z^2}{4}\right) + \frac{1 - I_0(\sqrt{a^2 z^2})}{a^2}\right]_0^1$$

$$= \left[\frac{1}{2} z^2 \log z \; {}_0\tilde{F}_1 \, ; \left(2 \, ; \frac{a^2 z^2}{4}\right)\right]_0^1 + \left[\frac{1 - I_0(\sqrt{a^2 z^2})}{a^2}\right]_0^1$$

$$= \left[\frac{1 - I_0(\sqrt{a^2})}{a^2} - \frac{1 - I_0(0)}{a^2}\right]$$

$$= \left[\frac{1 - I_0(\sqrt{a^2})}{a^2} - \frac{1 - 1}{a^2}\right]$$

$$= \frac{1 - I_0(a)}{a^2}.$$

Hence we have established the first assertion (2.1) of the theorem.

Next, to prove our second assertion (2.2), we make use the properties of Modified Bessel function of first kind, and apply little algebra, we have

$$\int_0^1 z^2 \log z \, I_0(az)\, dz$$

$$= \left[-\frac{1}{9} z^3 \left\{ {}_2F_3\left(\frac{3}{2}, \frac{3}{2}; 1, \frac{5}{2}, \frac{5}{2}; \frac{a^2 z^2}{4}\right) - 3 \log z \; {}_1F_2\left(\frac{3}{2}; 1, \frac{5}{2}; \frac{a^2 z^2}{4}\right)\right\}\right]_0^1$$

$$= \left[-\frac{1}{9} (1)^3 \left\{ {}_2F_3\left(\frac{3}{2}, \frac{3}{2}; 1, \frac{5}{2}, \frac{5}{2}; \frac{a^2 (1)^2}{4}\right) - 0\right\}\right]$$

$$= -\frac{1}{9} \; {}_2F_3\left(\frac{3}{2}, \frac{3}{2}; 1, \frac{5}{2}, \frac{5}{2}; \frac{a^2}{4}\right).$$

Hence we have established the first assertion (2.2) of the theorem.

Further, to prove our third assertion (2.3), we make use the properties of Modified Bessel function of first kind, and apply little algebra, we have

$$\int_0^1 z^3 \log z\, I_0(az)\, dz = \frac{1}{a^4}\Big[- a^2 z^2 I_2(az) + a^2 z^2 \log z\{2\, I_2(az) + az\, I_3(az)\}$$
$$+ 2az\, I_1(az) - 4I_0(az) + 4\Big]_0^1$$
$$= \frac{1}{a^4}\Big[- a^2 I_2(a) + 2a\, I_1(a) - 4\, I_0(a) + 4 + 4\, I_0(0) - 4\Big]$$
$$= \frac{1}{a^4}\Big[- a^2 I_2(a) + 2a\, I_1(a) - 4\, I_0(a) + 4\Big]$$
$$= \frac{-(a^2+4)I_0(a) + 4a\, I_1(a) + 4}{a^4}.$$

Hence we have established the first assertion (2.3) of the theorem.

Now we proceed, to prove our fourth assertion (2.4), we make use the properties of Modified Bessel function of first kind, and apply little algebra, we have

$$\int_0^1 z^4 \log z\, I_0(az)\, dz$$
$$= -\frac{1}{25}\left[z^5 \left\{ {}_2F_3\left(\frac{5}{2},\frac{5}{2};1,\frac{7}{2},\frac{7}{2};\frac{a^2z^2}{4}\right) - 5\log z\ {}_1F_2\left(\frac{5}{2};1,\frac{7}{2};\frac{a^2z^2}{4}\right)\right\}\right]_0^1$$
$$= -\frac{1}{25}\left[{}_2F_3\left(\frac{5}{2},\frac{5}{2};1,\frac{7}{2},\frac{7}{2};\frac{a^2}{4}\right) + 5\ {}_1F_2\left(\frac{5}{2};1,\frac{7}{2};0\right)\right]$$
$$= -\frac{1}{25}\ {}_2F_3\left(\frac{5}{2},\frac{5}{2};1,\frac{7}{2},\frac{7}{2};\frac{a^2}{4}\right).$$

Hence we have established our sixth assertion (2.4) of the theorem. The analogous proofs of the assertions (2.5) - (2.15) may be left as an exercise for the interested readers.

We thus have completed our proof of the above theorem.

3. Generalization of the Main Formulas:

In this section, we are recording generalization of the main formulas, which is given below;

$$\int_0^1 z^n \log z \, I_0(az) \, dz = - \frac{1}{(n+1)(n+1)} \, {}_2F_3\left(\frac{n+1}{2}, \frac{n+1}{2}; 1, \frac{n+3}{2}, \frac{n+3}{2}; \frac{a^2}{4}\right) \qquad (3.1)$$

Proof: We have established our proof for the Generalized formulas by inspecting one by one result.

4. Concluding Remarks and Observations:

In our present investigation, we have made use of the Bessel function as well as the hypergeometric and the generalized hypergeometric functions, with a view developing several definite integrals some integral involving Bessel functions of the first kind. The results derived in this article are believed to be new and would extend and unify those that are available in the scientific literature.

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$$1+\frac{\alpha}{1}\cdot\frac{\beta}{\gamma}x+\frac{\alpha(\alpha+1)}{1.2}\cdot\frac{\beta(\beta+1)}{\gamma(\gamma+1)}x^2+\ldots,$$

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