The Mathematics Education ISSN 0047-6269
Volume-LIX, No. 4, December 2025
Journal website: https://internationaljournalsiwan.com/Mathematic.php
ORCID Link: https://orcid.org/0009-0006-7467-6080
International Impact Factor: 7.75 https://impactfactorservice.com/home/journal/2295
Google Scholar: https://scholar.google.com/citations?hl=en&user=UOfM8B4AAAAJ
Refereed and Peer-Reviewed Quarterly Journal

On Computations of Definite Integrals Involving *G*-function

by **M.P. Chaudhary*,**
International Scientific Research and Welfare Organization, New Delhi-110018, India

Salahuddin, *Department of Mathematics, AMET University, Chennai-603112, India*

Hervé Wendpagnagdé Ouedraogo, *Department de Mathematics, Ecole Normale Superieure, Koudougou, Burkina Faso*

Sabahat Parveen, *Department of Mathematics, Maharashtra College of Arts, Science and Commerce, Mumbai-400008, India*

*Corresponding author: (e-mail: dr.m.p.chaudhary@gmail.com)

(Received: August 18, 2025; Accepted: October 3, 2025; Published Online: December 31, 2025)

Abstract:

In this article, authors developed six definite integrals involving G-function, and compute their numerical values. Several closely-related results such as (for example) Generalized hypergeometric functions are also considered. These results provid some extensions in the scientific literature. Furthermore, these integrals play a significant role in the applied Mathematics.

Keywords and Phrase: Bessel Function, Ramanujan's series.

2020 Mathematics Subject Classifications: 33B50, 33C05, 33C10, 33D50, 33D60, 33D67.

1. Introduction:

Bessel function of first kind of order n is defined as

$$J_n(\eta) = \frac{\left(\frac{\eta}{2}\right)^n}{\Gamma(n+1)}\, {}_0F_1\left(-\,;\, n+1\,;\, -\frac{\eta^2}{4}\right) \tag{1.1}$$

Seventeen Ramanujan's Series are [1, 4, 5]

$$R_4 \equiv \frac{4}{\pi} = 1 + \frac{7}{4}\left(\frac{1}{2}\right)^3 + \frac{13}{4^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{19}{4^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(6n+1)\left(\frac{1}{2}\right)_n}{4^n (n!)^3} \tag{1.2}$$

$$R_5 \equiv \frac{16}{\pi} = 5 + \frac{47}{64}\left(\frac{1}{2}\right)^3 + \frac{89}{64^2}\left(\frac{1.3}{2.4}\right)^3 + \frac{131}{64^3}\left(\frac{1.3.5}{2.4.6}\right)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(42n+5)\left(\frac{1}{2}\right)_n^3}{64^n (n!)^3} \tag{1.3}$$

$$R_6 \equiv \frac{32}{\pi} = (5\sqrt{5} - 1) + \left(\frac{47\sqrt{5} + 29}{64}\right)\left(\frac{1}{2}\right)^3\left(\frac{\sqrt{5} - 1}{2}\right)^8$$

$$+ \left(\frac{89\sqrt{5} + 59}{64^2}\right)\left(\frac{1.3}{2.4}\right)^3\left(\frac{\sqrt{5} - 1}{2}\right)^{16} + \cdots\cdots$$

$$= \sum_{n=0}^{\infty} \frac{(42\sqrt{5}\, n + 5\sqrt{5} + 30n - 1)\left(\frac{1}{2}\right)_n^3}{64^n (n!)^3}\left(\frac{\sqrt{5} - 1}{2}\right)^{8n} \tag{1.4}$$

$$R_7 \equiv \frac{27}{4\pi} = 2 + 17\,\frac{1}{2}\,\frac{1}{3}\,\frac{2}{3}\left(\frac{2}{27}\right) + 32\,\frac{1.3}{2.4}\,\frac{1.4}{3.6}\,\frac{2.5}{3.6}\left(\frac{2}{27}\right)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(15n+2)\left(\frac{1}{2}\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^3}\left(\frac{2}{27}\right)^n \tag{1.5}$$

$$R_8 \equiv \frac{15\sqrt{3}}{2\pi} = 4 + 37\,\frac{1}{2}\,\frac{1}{3}\,\frac{2}{3}\left(\frac{4}{125}\right) + 70\,\frac{1.3}{2.4}\,\frac{1.4}{3.6}\,\frac{2.5}{3.6}\left(\frac{4}{125}\right)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(33n+4)\left(\frac{1}{2}\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n}{(n!)^3}\left(\frac{4}{125}\right)^n \tag{1.6}$$

$$R_9 \equiv \frac{5\sqrt{5}}{2\pi\sqrt{3}} = 1 + 12\,\frac{1}{2}\,\frac{1}{6}\,\frac{5}{6}\left(\frac{4}{125}\right) + 23\,\frac{1.3}{2.4}\,\frac{1.7}{6.12}\,\frac{5.11}{6.12}\left(\frac{4}{125}\right)^2 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(11n+1)\left(\frac{1}{2}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}{(n!)^3}\left(\frac{4}{125}\right)^n \tag{1.7}$$

$$R_{10} \equiv \frac{85\sqrt{85}}{18\pi\sqrt{3}} = 8 + 141\,\frac{1}{2}\,\frac{1}{6}\,\frac{5}{6}\left(\frac{4}{85}\right)^3 + 274\,\frac{1.3}{2.4}\,\frac{1.7}{6.12}\,\frac{5.11}{6.12}\left(\frac{4}{85}\right)^6 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(133n+8)\left(\frac{1}{2}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}{(n!)^3}\left(\frac{4}{85}\right)^n \tag{1.8}$$

$$R_{11} \equiv \frac{4}{\pi} = \frac{3}{2} - \frac{23}{2^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{43}{2^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(20n+3)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\;2^{2n+1}} \tag{1.9}$$

$$R_{12} \equiv \frac{4}{\pi\sqrt{3}} = \frac{3}{4} - \frac{31}{3.4^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{59}{3^2.4^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n(28n+3)\left(\frac{1}{2}\right)_n\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(n!)^3\;4^{n+1}} \tag{1.10}$$

$$R_{13} \equiv \frac{4}{\pi} = \frac{23}{18} - \frac{283}{18^3}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{543}{18^5}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (260n + 23) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 \; (18)^{2n+1}} \tag{1.11}$$

$$R_{14} \equiv \frac{4}{\pi\sqrt{5}} = \frac{41}{72} - \frac{685}{5.72^3}\, \frac{1}{2}\, \frac{1.3}{4^2} + \frac{1329}{5^2.72^5}\, \frac{1.3}{2.4}\, \frac{1.3.5.7}{4^2.8^2} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (644n + 41) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\, 5n\, (72)^{2n+1}} \tag{1.12}$$

$$R_{15} \equiv \frac{4}{\pi} = \frac{1123}{882} - \frac{22583}{882^3}\, \frac{1}{2}\, \frac{1.3}{4^2} + \frac{44043}{882^5}\, \frac{1.3}{2.4}\, \frac{1.3.5.7}{4^2.8^2} - \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (21460n + 1123) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 \; (882)^{2n+1}} \tag{1.13}$$

$$R_{16} \equiv \frac{2\sqrt{3}}{\pi} = 1 + \frac{9}{9}\, \frac{1}{2}\, \frac{1.3}{4^2} + \frac{17}{9^2}\, \frac{1.3}{2.4}\, \frac{1.3.5.7}{4^2.8^2} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(8n + 1) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3\, 9^n} \tag{1.14}$$

$$R_{17} \equiv \frac{1}{2\pi\sqrt{2}} = \frac{1}{9} + \frac{11}{9^3}\, \frac{1}{2}\, \frac{1.3}{4^2} + \frac{21}{9^5}\, \frac{1.3}{2.4}\, \frac{1.3.5.7}{4^2.8^2} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(10n + 1) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 \; (9)^{2n+1}} \tag{1.15}$$

$$R_{18} \equiv \frac{1}{3\pi\sqrt{3}} = \frac{3}{49} + \frac{43}{49^2}\, \frac{1}{2}\, \frac{1.3}{4^2} + \frac{83}{49^2}\, \frac{1.3}{2.4}\, \frac{1.3.5.7}{4^2.8^2} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(40n + 3) \left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 \; 49^{2n+1}} \tag{1.16}$$

$$R_{19} \equiv \frac{2}{\pi\sqrt{11}} = \frac{19}{99} + \frac{299}{99^3}\, \frac{1}{2}\, \frac{1.3}{4^2} + \frac{579}{99^5}\, \frac{1.3}{2.4}\, \frac{1.3.5.7}{4^2.8^2} + \dots$$

$$= \sum_{n=0}^{\infty} \frac{(280n+19)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 \; 99^{2n+1}} \tag{1.17}$$

and

$$R_{20} \equiv \frac{1}{2\pi\sqrt{2}} = \frac{1103}{99^2} + \frac{27493}{99^6}\,\frac{1}{2}\,\frac{1.3}{4^2} + \frac{53883}{99^{10}}\,\frac{1.3}{2.4}\,\frac{1.3.5.7}{4^2.8^2} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(26390n+1103)\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(n!)^3 \; 99^{4n+2}} \tag{1.18}$$

Qureshi et al. derived the following formule[3]:

$${}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{47}{42};\ 1,\ 1,\ \frac{5}{42};\ \frac{1}{64}\right) = \frac{16}{5\pi} \tag{1.19}$$

$${}_4F_3\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{(375-8\sqrt{5})}{330};1,\ 1,\ \frac{(45-8\sqrt{5})}{330};\ \frac{(47-21\sqrt{5})}{128}\right) = \frac{(8+40\sqrt{5})}{31\pi} \tag{1.20}$$

$${}_4F_3\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{17}{15};\ 1,\ 1,\ \frac{2}{15};\ \frac{2}{27}\right) = \frac{27}{8\pi} \tag{1.21}$$

$${}_4F_3\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{37}{33};\ 1,\ 1,\ \frac{4}{33};\ \frac{4}{125}\right) = \frac{15\sqrt{3}}{8\pi} \tag{1.22}$$

$${}_4F_3\left(\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{12}{11};\ 1,\ 1,\ \frac{1}{11};\ \frac{4}{125}\right) = \frac{5\sqrt{15}}{6\pi} \tag{1.23}$$

and

$${}_4F_3\left(\frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{141}{133};\ 1,\ 1,\ \frac{8}{133};\ \frac{4}{85}\right) = \frac{85\sqrt{255}}{432\pi}. \tag{1.24}$$

Generalized hypergeometric function is written as [2]:

$${}_\alpha F_\beta \left[\begin{matrix} a_1, a_2, \ldots, a_\alpha\,; \\ b_1, b_2, \ldots, b_\beta\,; \end{matrix} \quad z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_\alpha)_k z^k}{(b_1)_k (b_2)_k \ldots (b_\beta)_k k!} \tag{1.25}$$

Meijer G-function is defined in the form of line integral as [1]:

$$G_{p,q}^{m,n}\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle|\ z\right) = \frac{1}{2\pi l}\int_L \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)}\, z^s\, ds \tag{1.26}$$

2. Definite Integrals associated with *G*-functions:

In this section, we establish set of six definite integrals associated with *G*-functions, as follows:

Theorem 1: Each of the following assertion holds true:

$$\int_0^\infty e^{-t}\, t^{\frac{5}{42}}\, {}_3F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\, 1,\, 1,\, \frac{5}{42};\, \frac{t}{64}\right)dt = \frac{16}{5\pi}\,\Gamma\left(\frac{47}{42}\right) \tag{2.1}$$

$$\int_0^\infty e^{-t}\, t^{3/22-4/(33\sqrt{5})}\, {}_3F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\, 1,\, 1,\, \frac{(45-8\sqrt{5})}{330};\, \frac{(47-21\sqrt{5})}{128}\,t\right)dt$$

$$= \frac{(8+40\sqrt{5})}{31\pi}\,\Gamma\left(\frac{(375-8\sqrt{5})}{330}\right) \tag{2.2}$$

$$\int_0^\infty e^{-t}\, t^{\frac{2}{15}}\, {}_3F_3\left(\frac{1}{2},\frac{1}{3},\frac{2}{3};\, 1,\, 1,\, \frac{2}{15};\, \frac{2t}{27}\right)dt = \frac{27}{8\pi}\,\Gamma\left(\frac{17}{15}\right) \tag{2.3}$$

$$\int_0^\infty e^{-t}\, t^{\frac{4}{33}}\, {}_3F_3\left(\frac{1}{2},\frac{1}{3},\frac{2}{3};\, 1,\, 1,\, \frac{4}{33};\, \frac{4t}{125}\right)dt = \frac{15\sqrt{3}}{8\pi}\,\Gamma\left(\frac{37}{33}\right) \tag{2.4}$$

$$\int_0^\infty e^{-t}\, t^{\frac{1}{11}}\, {}_3F_3\left(\frac{1}{2},\frac{1}{6},\frac{5}{6};\, 1,\, 1,\, \frac{1}{11};\, \frac{4t}{125}\right)dt = \frac{5\sqrt{15}}{6\pi}\,\Gamma\left(\frac{12}{11}\right) \tag{2.5}$$

and

$$\int_0^\infty e^{-t}\, \sqrt[133]{t^8}\, {}_3F_3\left(\frac{1}{2},\frac{1}{6},\frac{1}{5};\, 1,\, 1,\, \frac{8}{133};\, \frac{4t}{85}\right)dt = \frac{85\sqrt{255}}{432\pi}\,\Gamma\left(\frac{141}{133}\right) \tag{2.6}$$

provided that each member of the assertions (2.1) to (2.6) exists.

Proofs: In this section, we provide proofs for the assertions (2.1) to (2.6), as given below:

We first prove our assertion (2.1) as:

$$\int_0^\infty e^{-t}\, t^{\frac{5}{42}}\, {}_3F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};\, 1,\, 1,\, \frac{5}{42};\, \frac{t}{64}\right)dt$$

$$= -\Gamma\left(\frac{47}{42}\right)\frac{\left(\prod_{k=1}^{3}\Gamma(b_k)\right)\sum_{k=0}^{\infty}\frac{\left(\frac{1}{64}-t\right)^k G_{4,4}^{3,2}\left(\begin{matrix}1,\frac{5}{42}+k,1+k,1+k\\ \frac{1}{2}+k,\frac{1}{2}+k,\frac{1}{2}+k,\frac{47}{42}+k\end{matrix}\middle|-\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\Gamma(a_k)}$$

$$=\Gamma\left(\frac{47}{42}\right)\sum_{k=0}^{\infty}\frac{64^{-k}\left(\left(\frac{1}{2}\right)_k\right)^3\left(\frac{47}{42}\right)_k}{k!\left(\frac{5}{42}\right)_k\left((1)_k\right)^2}=\Gamma\left(\frac{47}{42}\right){}_4F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{47}{42};1,1,\frac{5}{42};\frac{1}{64}\right)$$

Now using (1.19), we obtain:

$$\int_0^\infty e^{-t}\,t^{\frac{5}{42}}\,{}_3F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,\frac{5}{42};\frac{t}{64}\right)dt=\Gamma\left(\frac{47}{42}\right)*\frac{16}{5\pi}=\frac{16}{5\pi}\Gamma\left(\frac{47}{42}\right).$$

This completes our demonstration of the first assertion (2.1).

Next, we prove of second assertion (2.2), as:

$$\int_0^\infty e^{-t}\,t^{3/22-4/(33\sqrt{5})}\,{}_3F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,\frac{(45-8\sqrt{5})}{330};\frac{(47-21\sqrt{5})}{128}\,t\right)dt$$

$$= -\Gamma\left(\frac{375-8\sqrt{5}}{330}\right)\frac{\left(\prod_{k=1}^{3}\Gamma(b_k)\right)\sum_{k=0}^{\infty}\frac{\left(\frac{(47-21\sqrt{5})}{128}-t\right)^k G_{4,4}^{3,2}\left(\begin{matrix}1,\frac{3}{22}-\frac{4}{33\sqrt{5}}+k,1+k,1+k\\ \frac{1}{2}+k,\frac{1}{2}+k,\frac{1}{2}+k,\frac{25}{22}-\frac{4}{33\sqrt{5}}+k\end{matrix}\middle|-\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\Gamma(a_k)}$$

$$=\Gamma\left(\frac{375-8\sqrt{5}}{330}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{(47-21\sqrt{5})}{128}\right)^k\left(\left(\frac{1}{2}\right)_k\right)^3\left(\frac{375-8\sqrt{5}}{330}\right)_k}{k!\left((1)_k\right)^2\left(\frac{(45-8\sqrt{5})}{330}\right)_k}$$

$$=\Gamma\left(\frac{375-8\sqrt{5}}{330}\right){}_4F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{(375-8\sqrt{5})}{330};1,1,\frac{(45-8\sqrt{5})}{330};\frac{(47-21\sqrt{5})}{128}\right)$$

Applying (1.20), we have:

$$\int_0^\infty e^{-t}\,t^{3/22-4/(33\sqrt{5})}\,{}_3F_3\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1,\frac{(45-8\sqrt{5})}{330};\frac{(47-21\sqrt{5})}{128}\,t\right)dt$$

$$= \Gamma\left(\frac{375 - 8\sqrt{5}}{330}\right) \frac{(8 + 40\sqrt{5})}{31\pi} = \frac{(8 + 40\sqrt{5})}{31\pi}\, \Gamma\left(\frac{375 - 8\sqrt{5}}{330}\right).$$

This completes our demonstration of the first assertion (2.2).

Next, we prove of second assertion (2.3), as:

$$\int_0^{\infty} e^{-t}\, t^{2/15}\, {}_3F_3\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}; 1, 1, \frac{2}{15}; \frac{2t}{27}\right) dt$$

$$= -\Gamma\left(\frac{17}{15}\right) \frac{\left(\prod_{k=1}^{3}\Gamma(b_k)\right) \sum_{k=0}^{\infty} \frac{\left(\frac{2}{27} - t\right)^k G_{4,4}^{3,2}\left(\begin{matrix} 1, \frac{2}{15} + k, 1 + k, 1 + k \\ \frac{1}{3} + k, \frac{1}{2} + k, \frac{2}{3} + k, \frac{17}{15} + k \end{matrix} \middle| -\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\Gamma(a_k)}$$

$$= \Gamma\left(\frac{17}{15}\right) \sum_{k=0}^{\infty} \frac{\left(\frac{2}{27}\right)^k \left(\frac{1}{3}\right)_k \left(\frac{1}{2}\right)_k \left(\frac{2}{3}\right)_k \left(\frac{17}{15}\right)_k}{k!\, \left(\frac{2}{15}\right)_k \left((1)_k\right)^2} = \Gamma\left(\frac{17}{15}\right) {}_4F_3\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{37}{33}; 1, 1, \frac{4}{33}; \frac{4}{125}\right).$$

Now using (1.21), we got:

$$\int_0^{\infty} e^{-t}\, t^{\frac{2}{15}}\, {}_3F_3\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}; 1, 1, \frac{2}{15}; \frac{2t}{27}\right) dt = \Gamma\left(\frac{17}{15}\right) \frac{27}{8\pi} = \frac{27}{8\pi}\, \Gamma\left(\frac{17}{15}\right).$$

This completes our demonstration of the first assertion (2.3).

Next, we prove of second assertion (2.4), as:

$$\int_0^{\infty} e^{-t}\, t^{4/33}\, {}_3F_3\left(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}; 1, 1, \frac{4}{33}; \frac{4t}{125}\right) dt$$

$$= -\Gamma\left(\frac{37}{33}\right) \frac{\left(\prod_{k=1}^{3}\Gamma(b_k)\right) \sum_{k=0}^{\infty} \frac{\left(\frac{4}{125} - t\right)^k G_{4,4}^{3,2}\left(\begin{matrix} 1, \frac{4}{33} + k, 1 + k, 1 + k \\ \frac{1}{3} + k, \frac{1}{2} + k, \frac{2}{3} + k, \frac{37}{33} + k \end{matrix} \middle| -\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\Gamma(a_k)}$$

$$= \Gamma\left(\frac{37}{33}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{4}{125}\right)^{k}\left(\frac{1}{3}\right)_k\left(\frac{1}{2}\right)_k\left(\frac{2}{3}\right)_k\left(\frac{37}{33}\right)_k}{k!\left(\frac{4}{33}\right)_k\left((1)_k\right)^2} = \Gamma\left(\frac{37}{33}\right){}_4F_3\left(\frac{1}{2},\frac{1}{3},\frac{2}{3},\frac{37}{33};1,1,\frac{4}{33};\frac{4}{125}\right).$$

Now using (1.22), we obtain:

$$\int_0^{\infty} e^{-t}\, t^{\frac{4}{33}}\, {}_3F_3\left(\frac{1}{2},\frac{1}{3},\frac{2}{3};1,1,\frac{4}{33};\frac{4t}{125}\right)dt = \Gamma\left(\frac{37}{33}\right)\frac{15\sqrt{3}}{8\pi} = \frac{15\sqrt{3}}{8\pi}\Gamma\left(\frac{37}{33}\right).$$

This completes our demonstration of the first assertion (2.4).

Next, we prove of second assertion (2.5), as:

$$\int_0^{\infty} e^{-t}\, t^{1/11}\, {}_3F_3\left(\frac{1}{2},\frac{1}{6},\frac{5}{6};1,1,\frac{1}{11};\frac{4t}{125}\right)dt$$

$$= -\Gamma\left(\frac{12}{11}\right)\frac{\left(\prod_{k=1}^{3}\Gamma(b_k)\right)\sum_{k=0}^{\infty}\dfrac{\left(\frac{4}{125}-t\right)^{k} G^{3,2}_{4,4}\left(\left.\begin{matrix}1,\frac{1}{11}+k,1+k,1+k\\ \frac{1}{6}+k,\frac{1}{2}+k,\frac{5}{6}+k,\frac{12}{11}+k\end{matrix}\right| -\frac{1}{t}\right)}{k!}}{\prod_{k=1}^{4}\Gamma(a_k)}$$

$$= \Gamma\left(\frac{12}{11}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{4}{125}\right)^{k}\left(\frac{1}{6}\right)_k\left(\frac{1}{2}\right)_k\left(\frac{5}{6}\right)_k\left(\frac{12}{11}\right)_k}{k!\left(\frac{1}{11}\right)_k\left((1)_k\right)^2} = \Gamma\left(\frac{12}{11}\right){}_4F_3\left(\frac{1}{2},\frac{1}{6},\frac{5}{6},\frac{12}{11};1,1,\frac{1}{11};\frac{4}{125}\right).$$

Now using (1.23), we have:

$$\int_0^{\infty} e^{-t}\, t^{\frac{1}{11}}\, {}_3F_3\left(\frac{1}{2},\frac{1}{6},\frac{5}{6};1,1,\frac{1}{11};\frac{4t}{125}\right)dt = \Gamma\left(\frac{12}{11}\right)\frac{5\sqrt{15}}{6\pi} = \frac{5\sqrt{15}}{6\pi}\Gamma\left(\frac{12}{11}\right).$$

This completes our demonstration of the first assertion (2.5).

Next, we prove of second assertion (2.6), as:

$$\int_0^{\infty} e^{-t}\, \sqrt[133]{t^8}\, {}_3F_3\left(\frac{1}{2},\frac{1}{6},\frac{1}{5};1,1,\frac{8}{133};\frac{4t}{85}\right)dt$$

$$= -\Gamma\left(\frac{141}{133}\right) \frac{\left(\prod_{k=1}^{3}\Gamma(b_k)\right)\sum_{k=0}^{\infty}\frac{\left(\frac{4}{85}-t\right)^k G_{4,4}^{3,2}\left(\begin{array}{c|c} 1, \frac{8}{133}+k, 1+k, 1+k & -\frac{1}{t} \\ \frac{1}{6}+k, \frac{1}{2}+k, \frac{5}{6}+k, \frac{141}{133}+k & \end{array}\right)}{k!}}{\prod_{k=1}^{4}\Gamma(a_k)}$$

$$= \Gamma\left(\frac{141}{133}\right)\sum_{k=0}^{\infty}\frac{\left(\frac{4}{85}\right)^k\left(\frac{1}{6}\right)_k\left(\frac{1}{2}\right)_k\left(\frac{5}{6}\right)_k\left(\frac{141}{133}\right)_k}{k!\left(\frac{8}{133}\right)_k\left((1)_k\right)^2} = \Gamma\left(\frac{141}{133}\right){}_4F_3\left(\frac{1}{2},\frac{1}{6},\frac{5}{6},\frac{141}{133};1,1,\frac{8}{133};\frac{4}{85}\right).$$

Now using (1.24), we obtain:

$$\int_0^{\infty} e^{-t}\sqrt[133]{t^8}\;{}_3F_3\left(\frac{1}{2},\frac{1}{6},\frac{1}{5};1,1,\frac{8}{133};\frac{4t}{85}\right)dt = \Gamma\left(\frac{141}{133}\right)\frac{85\sqrt{255}}{432\pi} = \frac{85\sqrt{255}}{432\pi}\Gamma\left(\frac{141}{133}\right).$$

This completes our demonstration of assertion (2.6).

This obviously completes our proof of Theorem 1.

3. Compliance with Ethical Standards:

Funding: This study was funded to M.P. Chaudhary by the National Board of Higher Mathematics (NBHM) of the Department of Atomic Energy (DAE) of the Government of India by its sanction letter (Ref. No. 02011/12/2020 NBHM (R.P.)/R D II/ 7867) dated 19 October 2020.

Conflict of Interest: Both the authors declares that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

References:

[1] B.C. Berndt : *Ramanujan's notebooks*. Parts.I, Springer-Verlag, New York, 1985.

[2] M.P. Chaudhary : *Certain Aspects of Special Functions and Integral Operators*, LAMBERT Academic Publishing, Germany, 2014.

[3] M.P. Chaudhary : *A simple solution of some integrals given by Srinivasa Ramanujan, Resonance:* J. Sci. Education (publication of Indian Academy of Science, Bangalore), **13**(9), 2008, 882-884.

[4] M.I. Qureshi, I.H. Khan and M.P. Chaudhary : *On Certain Hypergeometric Summation Theorems Motivated by the works of Ramanujan, Chudnovsky and Borwein.* Journal of Mathematics Research, **2**(2010), 196-202.

[5] S. Ramanujan : *Notebooks of Srinivasa Ramanujan*. Vol. II, Tata Institute of Fundamental Research, Bombay, 1957; Reprinted by Narosa, 1984, New Delhi.

[6] B.J. Venkatachala, V. Vinay and C.S. Yogananda : *Ramanujan's papers*. Prism Books Pvt. Ltd., Bangalore, Mumbai, 2000.

[7] H. Bateman and A. Erdélyi : *Higher Transcendental Functions*, Vol. I. New York, McGraw-Hill, 1953.

[8] M.P. Chaudhary : *Certain Aspects of Special Functions and Integral Operators*, LAMBERT Academic Publishing, Germany, 2014.

[9] M.I. Qureshi, I.H. Khan and M.P. Chaudhary : *On certain hypergeometric summation theorems motivated by the works of Ramanujan, Chudnovsky and Borwein,* Journal of Mathematics Research, **2**(2010), 196-202.

[10] S. Ramanujan : *Notebooks of Srinivasa Ramanujan*. Vol. II, Tata Institute of Fundamental Research, Bombay, 1957; Reprinted by Narosa, New Delhi, 1987.

[11] M.A. Shpot, M.P. Chaudhary and R.B. Paris : *Integrals of products of Hurwitz zeta functions and the Casimir effect in ϕ^4 field theories*, J. Classical Analysis, **9**(2) (2016), 99-115.

[12] B.J. Venkatachala, V. Vinay and C.S. Yogananda : *Ramanujan's papers*. Prism Books Pvt. Ltd., Bangalore and Mumbai, 2000.

[13] M.P. Chaudhary : *A simple solution of some integrals given by Srinivasa Ramanujan*, Resonance: J. Sci. Education (publication of Indian Academy of Science, Bangalore), **13**(9) (2008) 882-884.

[14] M.P. Chaudhary, M.L. Kaurangini, I.O. Kiymaz, U.M. Abubakar, and E. Ata : *Fractional integrations for the new generalized hypergeometric functions*, J. of Ramanujan Society of Mathematics and Mathematical Sciences, **10**(2)(2023), 77-100.

[15] M.L. Kaurangini, M.P. Chaudhary, U.M. Abubakar, I.O. Kiymaz and E. Ata : *On some applications of the new extended hypergeometric function*, Journal of Contemporary Applied Mathematics, **14**(1)(2024), 28-45.

[16] M.L. Kaurangini, M.P. Chaudhary, U.M. Abubakar, I.O. Kiymaz and E. Ata : *Fractional calculus and fractional kinetic equation involving extended hypergeometric function*, J. of Ramanujan Society of Mathematics and Mathematical Sciences, **11**(1)(2023), 159-174.

[17] Salahuddin and M.P. Chaudhary : *Some elliptical integrals associated with hypergeometric functions*, J. of Ramanujan Society of Mathematics and Mathematical Sciences, **11**(1)(2023), 107-114.

[18] M.L. Kaurangini, M.P. Chaudhary, U.M. Abubakar, I.O. Kiymaz and E. Ata : *On some extended special functions with bi-Fox-Wright function kernel*, J. of Ramanujan Society of Mathematics and Mathematical Sciences, **11**(1)(2023), 17-42.

[19] M.P. Chaudhary, Salahuddin, Hervé Wendpagnagdé Ouedraogo and Sama Arjika : *Certain definite integral in association with complete elliptic integral*, The Mathematics Education, **LVI**(4)(2022), 1-8.

[20] M.P. Chaudhary, Salahuddin and Sangeeta Chaudhary : *Computation of a summation formula associated with certain special function*, Malaya J. Mat., **3**(2)(2015), 131-142.