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Certain Definite Integral in Association with Complete Elliptic Integral

by **M.P. Chaudhary,**
International Scientific Research and Welfare Organization,
New Delhi - 110018, India
Corresponding author (e-mail : dr.m.p.chaudhary@gmail.com)

Salahuddin,
Department of Mathematics,
PDM University, Bahadurgarh - 124507, Haryana, India

Hervé Wendpagnagdé Ouedraogo,
Départment de Mathématiques, Ecole Normale Supérieure
Koudougou PoBox 376, Burkina Faso
&
Sama Arjika,
Department of Mathematics and Informatics,
University of Agadez, Niger

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Abstract :

In this paper we have developed certain definite integral involving elliptic integral of first and second kind.

Keywords and Phrases : Elliptic integral, Gauss hypergeometric function, Boolean algebra.

AMS Mathematical Subject Classifications 2020 : 33C75, 33E05, 03G05.

1. Introduction :

Yurry A. Brychkov (see [2], p.155-156(4.1.6)) has established the following formulae:

$$\int_0^a \frac{1}{\sqrt{(a^2-x^2)}} \sin^{-1}(bx)\, dx = \frac{1}{2}\, [Li_2(ab) + Li_2(-ab)] \tag{1.1}$$

where $[|\arg(1-a^2b^2)| < \pi]$.

$$\int_0^a \sqrt{(a^2-x^2)} \sin^{-1}(bx)\, dx = \frac{a}{4b}\left\{ \frac{1-a^2b^2}{2ab} \ln \frac{1+ab}{1-ab} + ab[Li_2(ab) - Li_2(-ab)] - 1 \right\} \tag{1.2}$$

where $[|\arg(1-a^2b^2)| < \pi]$.

$$\int_0^a x\sqrt{(a^2-x^2)} \sin^{-1}(bx)\, dx = \frac{a^2}{9b}\, [2(1-2a^2b^2)D(ab) - (1-3a^2b^2)K(ab)] \tag{1.3}$$

where $[|\arg(1-a^2b^2)| < \pi]$.

And

$$\int_0^a \frac{x}{\sqrt{(a^2-x^2)}} \sin^{-1}(bx)\, dx = a^2b[K(ab) - D(ab)] \tag{1.4}$$

where $[|\arg(1-a^2b^2)| < \pi]$.

A generalized hypergeometric function ${}_{\alpha}F_{\beta}(a_1,\dots, a_{\alpha}; b_1,\dots, b_{\beta}; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{\zeta+1}}{c_{\zeta}} = \frac{P(\zeta)}{Q(\zeta)} = \frac{(\zeta+a_1)(\zeta+a_2)\dots(\zeta+a_{\alpha})}{(\zeta+b_1)(\zeta+b_2)\dots(\zeta+b_{\beta})(\zeta+1)} z. \tag{1.5}$$

Where $\zeta + 1$ in the denominator is present for historical reasons of notation (see[6], p.12(2.9)) and the resulting generalized hypergeometric function is written as under:

$$ {}_{\alpha}F_{\beta}\left[\begin{matrix} a_1, a_2,..., a_\alpha\ ; \\ b_1, b_2,...., b_\beta\ ; \end{matrix}\ z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k .... (a_\alpha)_k\, z^k}{(b_1)_k(b_2)_k .... (b_\beta)_k\, k!} \tag{1.6} $$

where the parameters $b_1, b_2,...., b_\beta$ are positive integers.

The complete elliptic integral of the first kind K is defined as

$$ K(\eta) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - \eta^2 \sin^2\theta}} = \int_0^1 \frac{dt}{\sqrt{(1 - t^2)(1 - \eta^2 t^2)}} \tag{1.7} $$

In power series

$$ K(\eta) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \eta^{2n} = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[P_{2n}(0)\right]^2 \eta^{2n} \tag{1.8} $$

where P_n is the Legendre polynomial.

In terms of the Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$ K(\eta) = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \eta^2\right) \tag{1.9} $$

The complete elliptic integral of the second kind E is defined as

$$ E(\eta) = \int_0^{\frac{\pi}{2}} \sqrt{1 - \eta^2 \sin^2\theta}\ d\theta = \int_0^1 \frac{\sqrt{1 - \eta^2 t^2}}{\sqrt{1 - t^2}}\, dt \tag{1.10} $$

It can be expressed as a power series

$$ E(\eta) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left[\frac{(2n)!}{2^{2n}(n!)^2}\right]^2 \frac{\eta^{2n}}{1 - 2n} \tag{1.11} $$

In terms of the Gauss hypergeometric function

$$ E(\eta) = \frac{\pi}{2}\ {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; \eta^2\right) \tag{1.12} $$

The fundamental operations of Boolean algebra are as follows:

(A). AND (conjunction), denoted $\xi \wedge \omega$, satisfies $\xi \wedge \omega = 1$ if $\xi = \omega = 1$, and $\xi \wedge \omega = 0$ otherwise.

(B). OR (disjunction), denoted $\xi \vee \omega$, satisfies $\xi \vee \omega = 0$ if $\xi = \omega = 0$, and $\xi \vee \omega = 1$ otherwise.

(C). NOT (negation), denoted $\neg \xi$, satisfies $\neg \xi = 0$ if $\xi = 1$ and $\neg \xi = 1$ if $\xi = 0$.

2. Main Formulae of the Integration :

In this section, we establish a set of twelve new integral formulaes:

$$\int_0^a x\sqrt{a^2-x^2}\tan^{-1}(cx)dx = \frac{\pi[a^2c^2(2\sqrt{a^2c^2+1}-3)+2(\sqrt{a^2c^2+1}-1)]}{12c^3} \tag{2.1}$$

where $a > 0$.

$$\int_0^a x\sqrt{a^2-x^2}\tanh^{-1}(cx)dx = \frac{\pi[a^2c^2(-2\imath\sqrt{a^2c^2-1}+3)+2\imath(\sqrt{a^2c^2-1}-2)]}{12c^3} \tag{2.2}$$

where $c < 0 \wedge ac < -1$.

$$\int_0^a \frac{\sqrt{a^2-x^2}\tan^{-1}(cx)}{x}dx = \frac{\pi(-\sqrt{a^2c^2+1}+ac\sinh^{-1}(ac)+1)}{2c} \tag{2.3}$$

where $c \in \mathbb{R} \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0$.

$$\int_0^a x\sqrt{a^2-x^2}\sinh^{-1}(cx)dx$$
$$= \frac{(3a^4c^4+5a^2c^2+2)K(-a^2c^2)-2(2a^2c^2+1)E(-a^2c^2)}{9c^3} \tag{2.4}$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

$$\int_0^a x^3\sqrt{a^2-x^2}\sinh^{-1}(cx)dx = \frac{1}{225c^5}[(a^2c^2+1)(30a^4c^4-7a^2c^2-24)K(-a^2c^2)$$
$$+(-31a^4c^4+19a^2c^2+24)E(-a^2c^2)] \tag{2.5}$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

$$\int_0^a x^5 \sqrt{a^2-x^2} \sinh^{-1}(cx)dx = \frac{1}{11025c^7} [(a^2c^2+1)(840a^6c^6 - 241a^4c^4+48a^2c^2 + 720)K(-a^2c^2) - 2(389a^6c^6 - 176a^4c^4 + 204a^2c^2 + 360)E(-a^2c^2)] \quad (2.6)$$

where $a > 0 \wedge c > 0$.

$$\int_0^a \frac{\sqrt{a^2-x^2} \sinh^{-1}(cx)}{x} dx = \frac{1}{4} \pi a^2 c \; {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} ; \frac{3}{2}, 2; -a^2c^2\right) \quad (2.7)$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

$$\int_0^a \frac{x \sinh^{-1}(cx)}{\sqrt{a^2-x^2}} dx = \frac{(a^2c^2+1)K(-a^2c^2) - E(-a^2c^2)}{c} \quad (2.8)$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

$$\int_0^a \frac{x^3 \sinh^{-1}(cx)}{\sqrt{a^2-x^2}} dx = \frac{(-5a^2c^2+2)E(-a^2c^2) + (3a^4c^4+2a^2c^2-1) K(-a^2c^2)}{9c^3} \quad (2.9)$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

$$\int_0^a \frac{x^5 \sinh^{-1}(cx)}{\sqrt{a^2-x^2}} dx = \frac{1}{225c^5} [(-94a^4c^4 + 31a^2c^2 - 24)E(-a^2c^2) + (120a^6c^6 + 77a^4c^4 - 19a^2c^2 + 24)K(-a^2c^2)] \quad (2.10)$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

$$\int_0^a \frac{x^7 \sinh^{-1}(cx)}{\sqrt{a^2-x^2}} dx = \frac{1}{3675c^7} [(-1276a^6c^6 + 389a^4c^4 - 256a^2c^2 + 240) E(-a^2c^2) + 2(840a^8c^8 + 529a^6c^6 - 123a^4c^4 + 68a^2c^2 - 120)K(-a^2c^2)] \quad (2.11)$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

$$\int_0^a \frac{x^9 \sinh^{-1}(cx)}{\sqrt{a^2-x^2}}\,dx = \frac{1}{99225c^9}\,[(a^2c^2+1)(40320a^8c^8 - 15208a^6c^6 + 9549a^4c^4 - 6600a^2c^2 + 4480)\,K(-a^2c^2) + (-30064a^8c^8 + 8776a^6c^6 - 5409a^4c^4 + 4360a^2c^2 - 4480)\,E(-a^2c^2)] \tag{2.12}$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

$$\int_0^a \frac{x^{11} \sinh^{-1}(cx)}{\sqrt{a^2-x^2}}\,dx = \frac{1}{2401245c^{11}}\,[2(a^2c^2+1)(a^2c^2(443520a^8c^8 - 169304a^6c^6 + 108693a^4c^4 - 78081a^2c^2 + 57344) - 40320)\,K(-a^2c^2) + (a^2c^2(-653344a^8c^8 + 185512a^6c^6 - 110241a^4c^4 + 83698a^2c^2 - 743368) + 80640)E(-a^2c^2)] \tag{2.13}$$

where $\mathrm{Re}(c) \geq 0 \wedge \mathrm{Re}(a) > 0 \wedge \mathrm{Im}(a) = 0 \wedge (\mathrm{Im}(c) > 0 \vee \mathrm{Re}(c) > 0)$.

3. Derivation of Main Formulae :

Proof of (2.1) :

$$\int_0^a x\sqrt{a^2-x^2}\,\tan^{-1}(cx)dx$$

$$= \Bigg[-\frac{1}{6c^3}\left(2c^3(a^2-x^2)^{3/2}\tan^{-1}(cx) + c^2x\sqrt{a^2-x^2} + (3a^2c^2+2)\tan^{-1}\left(\frac{x}{\sqrt{a^2-x^2}}\right)\right)$$

$$+ \iota(a^2c^2+1)^{3/2}\log\left(\frac{12c^4(-\iota\sqrt{a^2c^2+1}\sqrt{a^2-x^2} - \iota a^2c + x)}{(a^2c^2+1)^{5/2}(cx+\iota)}\right)$$

$$- \iota(a^2c^2+1)^{3/2}\log\left(\frac{12c^4(\iota\sqrt{a^2c^2+1}\sqrt{a^2-x^2} + \iota a^2c + x)}{(a^2c^2+1)^{5/2}(cx-\iota)}\right)\Bigg]_0^a$$

applying the properties of definit integral we have,

$$= \frac{\pi[a^2c^2(2\sqrt{a^2c^2+1} - 3) + 2(\sqrt{a^2c^2+1} - 1)]}{12c^3}, \qquad \text{where } a > 0.$$

We thus have completed our proof of (2.1).

Proofs of (2.2) - (2.13) : The proofs of (2.2) - (2.13) are on the similer pattern to the proof of (2.1). Hence, we left the proof for the readers.

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Conflicts of Interest : All the four authors declare that they have no conflict of interest.

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