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Subsequential Convergence in Locally Convex Spaces

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Abstract :

Given a locally convex Hausdroff linear topological space, we will construct the finest Locally convex topology with the same sequentially compact sets as the initial topology.

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1. Introduction :

It is evergreen problem in topology whether it is possible to construct a finest topology without disturbing the character of initial topology. Such a problem was studied by J.H. Webb[6], which prompted us to construct a finest locally convex topology, which has the same sequentially compact sets as the initial topology. Here it may be mentioned that pre-compact sets need not be sequentially compact but converse is true and hence our construction is a more general version of J.H. Webb. We have also examined completeness of $X^{\oplus}$ with respect to topology generated by class of bounded subsets of X.

By X we designate a fixed locally convex topological vector space.

X^* = Algebraic dual of X

$X = (X, \tau)'$ = Topological dual of X.

$$X^b = (X, \tau)^b = \{f \in X^* : \sup_{x \in B} |f(x)| < \infty \text{ for all } \tau\text{-bounded } B \subset X\}$$

$X^+ = (X, \tau)^+$ = sequential dual of X

$$= \{f \in X^* : f(x) \to 0 \text{ for all } \tau \text{ null } \{x_n\}\}$$

$X^\oplus = (X, \tau)^\oplus$ = subsequential dual of X

$$= \{f \in X^* \, |f(x_{nk})| \to 0 \text{ for all } \tau \text{ null } \{x_{nk}\}\}$$

Then clearly $X' \subset X^+ \subset X^\oplus \subset X^b \subset X^*$.

2. Sequentially Compact :

A set M is called sequentially compact if every sequence of points of M contains a subsequence which is convergent to a point of M.

Now, we will construct the finest locally topology on X with the same sequentially compact sets as the initial topology τ, and e will denote this by τ^s. Two constructions are possible one internal and other external.

2.1 Proposition : Let (X, τ) be a locally convex space and S denote the class of all τ-sequentially compact subsets of X. Let U be the class of all absolutely convex subsets of X, with the condition that $\forall\, S \in S$, there exist finite number of points $x_1, x_2, \ldots, x_n$ in S such that

$S \subseteq \cup_{i=1}^{n} (x_i + V)$ then,

(i) $V \in U \Rightarrow \alpha V \in U, \forall \alpha$

(ii) Every $V \in U$ is absorbent

(iii) U is closed under finite intersection.

Proof : (i) As $S \subseteq \cup_{i=1}^{n} (x_i + V)$ then since S is τ-sequential compact subset therefore αS is also τ-sequentially compact. Hence $\alpha S \subseteq \cup_{i=1}^{n} (x_i + \alpha V)$.

Which implies that $\alpha V \in U$.

(ii) Let $V \in U$ and $x \in X$, consider the set $A = \{\alpha x : |\alpha| \leq 1\}$ which is a member of S. A is contained in $\cup_{i=1}^{n} (\alpha_i x + V)$ where $\alpha_1, ..., \alpha_n$ are scalars such that $|\alpha_i| \leq 1$, i.e., $A \subseteq \cup_{i=1}^{n} (\alpha_i x + V)$. If α_0 is any other scalar such that $|\alpha_0| \leq 1$ and $\alpha_0 \neq \alpha_i$ (for $i = 1$ to n). Then $\alpha_0 x \in A$.

So, $\quad \alpha_0 x \in \cup_{i=1}^{n} (\alpha_i x + V)$

$\Rightarrow \quad \alpha_0 x \in (\alpha_j x + V)$ for some j

$\Rightarrow \quad \alpha_0 x - \alpha_j x \in V$

$\Rightarrow \quad x \in (\alpha_j - \alpha_0)^{-1} V$, As V is balanced so V absorbs x.

(iii) Let V_1 and $V_2 \in U$, then $V_1 \cap V_2$ is absolutely convex. Let $S \in S$, then there exists points $x_1, x_2, ..., x_m, y_1, y_2, ..., y_n$ in S such that,

$$S \subseteq \cup_{i=1}^{n} (x_i + V_1) \text{ and } S \subseteq \cup_{j=1}^{n} (y_j + V_2),$$

So, $\quad S \subseteq \cup_{i,j} (x_i + V_1) \cap (y_j + V_2)$

Now $z_{ij} \in S \cap (x_i + V_1) \cap (y_j + V_2)$ for some i and j

$$\Rightarrow \quad z_{ij} - x_i \in V_1 \text{ and } z_{ij} - y_j \in V_2 \qquad (1)$$

If $\quad z \in (x_i + V_1) \cap (y_j + V_2)$

$$\Rightarrow \quad z - x_i \in V_1 \text{ and } z - y_j \in V_2 \qquad (2)$$

as V_1 and V_2 are absolutely convex thus,

$\Rightarrow \quad z - z_{ij} \in 2V_1$ and $z - z_{ij} \in 2V_2$

$\Rightarrow \quad z - z_{ij} \in 2(V_1 \cap V_2)$

$\Rightarrow \quad z \in z_{ij} + 2(V_1 \cap V_2)$

$\Rightarrow \quad S \subseteq \cup_{i,j}(z_{ij} + 2(V_1 \cap V_2))$, so $(V_1 \cap V_2) \in U$

Take U as a base at 0 for a locally topology on X, which is denoted by τ^s, then $\tau \leq \tau^s$. Now by definition every τ^s - sequentially compact subset of X is τ - sequentially compact and every τ - sequentially compact subset of X is τ^s - sequentially compact, so the topologies τ and τ^s have the same sequentially compact sets on X.

3. External Construction of τ^s :

3.1 Proposition : Let $< X , Y>$ be a dual pair of linear spaces. Let δ be a class of $\sigma(X, Y)$ - closed subsets of X generating on Y. The topology τ_n of uniform convergence of members of η, then following statements are equivalent.

(a) Each $M \in \delta$ is τ_n - sequentially compact in X.

(b) Each $N \in \eta$ is τ_δ - sequentially compact in Y.

Proof : It immediately follows from the fact that sequentially compact set is closed.

3.2 Lemma : Let (X, τ) be a locally convex space, then

(i) $X^c = \{f \in X^* : \sup_{x \in S} |f(x)| < \infty \text{ for every } S \in S\}$

(ii) If τ_1 is another locally convex topology on X such that τ and τ_1 have the same sequentially compact sets in X then τ and τ_1 have the same closed sets in X.

Proof : (i) Every mackey convergent sequence is sequentially compact sets in X, then τ and τ_1 have the same closed sets in X.

(ii) By (i), $(X, \tau)^c = (X, \tau_1)^c$ hence $\tau^c = \tau_1^c = \mu(X, X^c)$.

So τ and τ_1 have the same closed sets in X.

3.3 Definition : If (X, τ) is a locally convex space, we will denote by τ^0 the topology on X^c of uniform convergence on the τ - sequentially compact subsets of X.

3.4 Proposition : If (X, τ) is a locally convex space then $\tau^s = \tau^{00}$, where τ^{00} denotes the topology on X of uniform convergence on the τ^0 - sequentially compact subset of X^c.

Proof : In the proposition 3.1, take $Y = X^c$ i.e. consider dual pair (X, X^c) and $\delta = S$ and η the class of τ - equicontinuous subsets of X'. Then $\tau = \tau_\eta$ and $\tau^0 = \tau_\delta$. So every equicontinuous subsets of X' is τ^0 - sequentially compact. Then $\tau \le \tau^{00}$.

Now take $Y = X^c$ and $\delta = S$ and η the class of τ^0 - sequentially compact subsets of X^c, so every τ - sequentially compact subsets of X is τ^{00} - sequentially compact.

So τ and τ^{00} have the same sequentially compact sets in X.

Now we will show τ^{00} is the finest topology. Let τ_1 is another locally convex topology on X with the same sequentially compact sets as τ. Then by Lemma 3.2 (ii) $(X, \tau)^c = X^c$.

On X^c both the topologies τ^0 and τ_1^0 coincides. So τ^{00} and τ_1^{00} but $\tau_1 \le \tau_1^{00}$, so $\tau_1 \le \tau^{00}$. This shows that τ^{00} is the finest locally convex topology on X with the same sequentially compact sets as τ. Hence $\tau^{00} = \tau^s$.

4. Completeness :

Now we will examine completeness of $X^{\oplus}$ with respect to topology generated by the class of bounded subsets of X.

Let γ be a family of bounded subsets of X, such that $\cup \{A : A \in \gamma\}$ spans X. The uniform convergence topology on X^b on the members of γ will be denoted by τ_γ.

ν = the class of sequences in X which are mackey convergent to 0.

λ = the class of τ - null subsequences of a sequence in X.

4.1 Proposition : If $\nu \le \gamma$ then (X^b, τ_γ) is complete.

It is by [6] proposition 3.1.

4.2 Proposition : If $\lambda \leq \gamma$ then $X^{\oplus}$ is a τ_γ - closed subspace of X^b.

Proof : Let $f \in X^{\oplus}$, which is τ_γ - closure of $X^{\oplus}$ in X^b, consider a τ - null subsequence of a sequence in X, then there exists $A \in \gamma$ such that $\subseteq A$ since $f \in X^{\oplus}$ so for any $\varepsilon > 0$ there is $g \in X^{\oplus}$ such that $f - g$ is a member of $(\varepsilon/2)\,\acute{A}$.

Now choose N such that $|< x_n, g >| < \varepsilon/2$ for all $n > N$. $|< x_n, f >| \leq |< x_n, f - g >| + |< x_n, g >|$.

So $|< x_n, f >| < \varepsilon$ for all $n > N$. Hence $f \in X^{\oplus}$. So $X^{\oplus}$ is τ_γ - closed in X^b.

4.3 Theorem : If $\lambda \leq \gamma$ then $X^{\oplus}$ is a τ_γ - complete subspace of X^b.

Proof : It immediately follows above proposition.

4.4 Definition : Let (X, τ) be locally convex Hausdroff space. We define the family ξ_r in X to be the set of all subsets A of X such that every sequence of points A has a Cauchy subsequence. We define the topology τ^g on X^b to be the topology of uniform convergence on the member of ξ_r.

Define the topology τ^n on X^b to be the topology of uniform convergence on the class of all τ - null subsequences of a sequence in X.

Define the topology τ^0 on X^c to be the topology of uniform convergence on τ - sequentially compact subsets of X.

4.5 Theorem : $X^{\oplus}$ is complete under each of the topologies τ^n, τ^g, τ^0 and $\beta(X^{\oplus}, X)$, (where β denotes the strong topology).

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