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Central SS-Elements of a Ring

by **Krishna Kumar Srivastava,** *Department of Mathematics, Government P.G. College, Hamirpur - 210301, India*

Abstract :

The notions of Central SS-elements, weakly idempotent SS-element of a ring are initiated and some results are obtained. Further, it is shown that every element 'a' of ring has a decomposition $a = e + f$, where e is an idempotent element and f is weakly idempotent SS-element satisfying the condition $ef = 0$.

Keywords : Ring, idempotent element, decomposition, SS-element

AMS Subject Classification No : 16A70(1991).

Introduction :

In [8], W.B. Vasantha Kandaswamy initiated the notions of SS-element of a ring and SS-element and obtained several results. In [1], we have proved that a Boolean algebra, regarded as a ring, is not an SS-ring. In [6],we have characterized SS-elements of a ring and applied the notion of SS-element to enumerate the number of non-zero divisors of zero of a class of a group algebras and to examine the semi simplicity of a class of group algebras. In [7], we have shown that the class of SS-rings contains a class of Foster's Boolean-like rings properly. In this we initiate the notions of central SS-element and weakly idempotent SS-element of a ring and obtain some results. Further, it is shown that every element a of a 3-ring has a decomposition $a = e + f$, where e is an idempotent element and f is weakly idempotent SS-element satisfying the condition $ef = 0$.

In section -1, we give the definitions of SS-element, SS-ring and 3-ring with examples. For the sake of completeness, we give some theorems. In section-2, we define central SS-element and weakly idempotent SS-element of a ring. Further, we state and prove our results in this section. Examples are provided to illustrate our notions. Basic concepts of ring theory are taken from [2]. In this paper ring means ring with unity. We begin with the following.

Section-1 :

Definition 1.1 : An element a in a ring R is called an SS-element if $a^2 = a + a$. In a ring, there can be found always, two SS-elements 0 (additive identity), 2 (=1+1) which are called trivial SS-elements of the ring. An SS-element other than 0 and 2 is termed non-trivial SS-element.

Definition 1.2 : A ring R is said to be an SS-ring if it contains at least one non-trivial SS-element. Examples of SS-elements and SS-rings can be found naturally in the literature.

Example 1.3 : Let $G = \{e, a\}$ be a cyclic group of order 2 and let $Z_2 = \{0, 1\}$ be a field of characteristic 2. $Z_2(G) = \{0, a, e, e + a\}$ is a group algebra with respect to the operations defined by the table 1.1 and 1.2.

Table 1.1

$+$	0	a	e	$e+a$
0	0	a	e	$e+a$
a	a	0	$e+a$	e
e	e	$e+a$	0	a
$e+a$	$e+a$	e	a	0

Table 1.2

$\cdot$	0	a	e	$e+a$
0	0	0	0	0
a	0	e	e	$e+a$
e	0	a	e	$e+a$
$e+a$	0	$e+a$	$e+a$	0

We can easily see that the element $e + a$ is a non-trivial SS-element of the group algebra $Z_2(G)$ and hence $Z_2(G)$ is an SS-ring.

Example 1.4 : Let $Z_3 = \{0, 1, 2\}$ be a prime field of characteristic 3 and let $G = \{g : g^2 + 1\}$ be a group. $Z_3(G) = \{0, 1, 2, g, 1 + 2g, 2 + 2g\}$ is a group algebra with respect to the operations defined by the table 1.3 and 1.4.

Table 1.3

$+$	0	1	2	g	$2g$	$1+g$	$2+g$	$1+2g$	$2+2g$
0	0	1	2	g	$2g$	$1+g$	$2+g$	$1+2g$	$2+2g$
1	1	2	0	$1+g$	$1+2g$	$2+g$	g	$2+2g$	$2g$
2	2	0	1	$2+g$	$2+2g$	g	$1+g$	$2g$	$1+2g$
g	g	$1+g$	$2+g$	$2g$	0	$1+2g$	$2+2g$	1	2
$2g$	$2g$	$1+2g$	$2+2g$	0	g	1	2	$1+g$	$2+g$
$1+g$	$1+g$	$2+g$	g	$1+2g$	1	$2+2g$	$2g$	2	0
$2+g$	$2+g$	g	$1+g$	$2+2g$	2	$2g$	$1+2g$	0	1
$1+2g$	$1+2g$	$2+2g$	$2g$	1	$1+g$	2	0	$2+g$	g
$2+2g$	$2+2g$	$2g$	$1+2g$	2	$2+g$	0	1	g	$1+g$

Table 1.4

$\cdot$	0	1	2	g	$2g$	$1+g$	$2+g$	$1+2g$	$2+2g$
0	0	0	0	0	0	0	0	0	0
1	0	1	2	g	$2g$	$1+g$	$2+g$	$1+2g$	$2+2g$
2	0	2	1	$2g$	g	$2+2g$	$1+2g$	$2+g$	$1+g$
g	0	g	$2g$	1	2	$1+g$	$1+2g$	$2+g$	$2+2g$
$2g$	0	$2g$	g	2	1	$2+2g$	$2+g$	$1+2g$	$1+g$
$1+g$	0	$1+g$	$2+2g$	$1+g$	$2+2g$	$2+2g$	0	0	$1+g$
$2+g$	0	$2+g$	$1+2g$	$1+2g$	$2+g$	0	$2+g$	$1+2g$	0
$1+2g$	0	$1+2g$	$2+g$	$2+g$	$1+2g$	0	$1+2g$	$2+g$	0
$2+2g$	0	$2+2g$	$1+g$	$2+2g$	$1+g$	$1+g$	0	0	$2+2g$

We can easily see that the elements $1+g$, $1+2g$ are non-trivial SS-elements of the group algebra $Z_3(g)$ and hence it is an SS-ring.

Theorem 1.5 (See [1]) : If a non-zero element a of a ring R is an SS-element, then it should necessarily satisfy the condition $a^2 \neq a$.

Theorem 1.6 (See [1]) : A Boolean algebra, regarded as a ring, is not an SS-ring.

Definition 1.7 : A ring R is a 3-ring if $a^3 = a$ and $3a = 0$ for every a in R.

Section - 2 :

Definition 2.1 : An SS-element a of a ring R is said to be a central SS-element if $ax = xa$, for all $x \in R$.

Example 2.2 : The SS-element $e + a$ of the SS-ring in example 1.3 is a central SS-element. The SS-elements $1+g$, $1+2g$, of the SS-ring in example 1.4 are central SS-elements as well.

Definition 2.3 : Two SS-elements a, b of a ring R are said to be orthogonal if $ab = 0$. Like wise, the SS-elements $e_1, e_2, \ldots, e_n$ are orthogonal if $e_i e_j = 0$ for all i, j with $i \neq j$.

Example 2.4 : The SS-elements $1+g$, $1+2g$ of the SS-ring in example 1.4 are orthogonal.

Definition 2.5 : An SS-element a of a ring R is said to be weakly idempotent if $a^4 = a^2$.

Example 2.6 : The SS-elements $1+g$, $1+2g$ of the group algebra in example 1.4 are weakly idempotent SS-elements.

In view of theorem 1.5, it is evident that the notions of idempotent element and SS-element of a ring are two opposite extreme ends but the product of an idempotent element and a central SS-element in an SS-element as well. We prove this in the following theorem.

Theorem 2.7 : In a ring R the product of an idempotent element and a central SS-element is an SS-element.

Proof : If r is an idempotent element and a is a central SS-element then $r^2 = r$ and $a^2 = a + a$, $ax = xa$, for all $x \in R$. Now,

$$(ra)^2 = rara = r(ar)a = r^2a^2 = r^2(a+a) = r(a+a) = ra+ra.$$

Hence, ra is an SS-element of R.

Theorem 2.8 : If $e_i (i = 1,..., n)$ are orthogonal idempotent elements and a is a central SS-element of a ring then the element $u = \sum_{i=1}^{n} e_i\, a$ is an SS-element.

Proof : $u^2 = \sum_{i=1}^{n} e_i a \sum_{i=1}^{n} e_i a = \sum_{i=1}^{n} e_i^2 a^2 = \sum_{i=1}^{n} e_i a + \sum_{i=1}^{n} e_i a = u + u$, since a is central SS-element and $e_j e_k = 0$, $j \neq k$, for all j, k $(=1,..., n)$. Hence, u is an SS-element.

Theorem 2.9 : In a ring R with idempotent element, every central SS-element, if exists, has more than one decomposition.

Proof : Let a be a central SS-element and let e be an idempotent element of the ring R. In view of the theorem 2.7, we have that the elements ae and ea are SS-elements of R. Write

$$a = ea + (a - ea).$$

Now, $(a - ea)^2 = (a - ea)(a - ea) = a^2 - ea^2 - ea^2 + ea^2 = a^2(1 - e) = (a + a)(1 - e) = a + a - ae - ae = (a - ae) + (a - ae)$. Moreover, $ea(a - ae) = eaa - eaea = ea^2 - eeaa = ea^2 - ea^2 = 0$.

Consequently, the elements ea, $a - ea$ are orthogonal SS-elements. Further, it is obvious that $1 - e \neq e$ and $1 - e$ is an idempotent element as $(1 - e)(1 - e) = 1 - e - e + e^2 = 1 - e$. In view of theorem 2.7, the element $a(1 - e)$ is an SS-element as well.

Now, in a second way, write $a = eae + ea(1 - e) + (1 - e)a(1 - e)$.

Again, in view of theorem 2.7, the elements eae, $ea(1 - e)$, $(1 - e)a(1 - e)$ are SS-elements. After simple calculations and in view of definition 2.3 the elements eae, $ea(1 - e)$, $(1 - e)a(1 - e)$ are orthogonal. This completes the proof of our assertion.

Finally, we state and prove a theorem on 3 ring.

Theorem 2.10 : In a 3-ring R every element a has a decomposition $a = e + f$, where e is an idempotent element and f is weakly idempotent SS-element.

Proof : It is obvious that $a^3 = a$, $3a = 0$. Write $a = e + f$, where $e = 2a - a^2$, $f = a^2 - a$. Now, $e^2 = (2a - a^2)(2a - a^2) = 4a^2 - 2a^3 - 2a^3 + a^4 = 2a^2 - a = -a^2 - (-2a) = 2a - a^2 = e$.

So, e is an idempotent element of R.

Next, $f^2 = (a^2 - a)(a^2 - a) = a^4 - a^3 - a^3 + a^2 = a - a^2$. On the other hand, $f + f = a^2 - a + a^2 - a = a - a^2$. So, f is an SS-element of R. The elements e and f are orthogonal as $ef = (2a - a^2)(a^2 - a) = 2a^3 - 2a^2 - a^4 + a^3 = 3a - 3a^2 = 0$.

Finally, $f^4 = (a - a^2)(a - a^2) = a^2 - a^3 - a^3 + a^4 = 2a^2 - 2a^3 = a - a^2 = f^2$. So, f is weakly idempotent SS-element. This completes the proof.

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