The Mathematics Education **ISSN 0047-6269**
Volume - LV, No. 1, March 2021
Refereed and Peer-Reviewed Quarterly Journal
Journal website : www.internationaljournalsiwan.com

A Brief Study of Projective Limit in a Function Space

by **Karuna Kumari Sharma,** *Research Scholar,*
Jai Prakash University, Chapra - 841301
E-mail : karunaksharma24@gmail.com
&
Lal Babu Singh, *Retd. Associate Professor,*
Department of Mathematics,
Jai Prakash University, Chapra - 841301
E-mail : lbsdmjpu@gmail.com

Abstract :

In this paper, we deal with projective limit, projective limit has been defined. It has been defined when $\alpha(f)$ is regular under a defined convergence and also when convergence closed under a defined convergence. We also define when $\alpha(f)$ is limit-closed under some convergence.

It has been proved that $\alpha^*(f)$ is limit-closed under $\alpha^*(f)\ \alpha(f)$ convergence. We defined when $f_\lambda(x)$ is a section of $\psi(x)$ and show that every function in $\alpha(f)$ is an $\alpha(f)$-limit of its sections.

Keywords : convergence-closed, limit-closed, projective limit, regular.

1. Introduction :

A function $\psi(x)$, in $\alpha(f)$ or outside $\alpha(f)$, is said to be **projective limit** (p-limit) of $f_\lambda(x)$ in $\alpha(f)$ relative to $\beta(f)$, and we write $\psi(x) = \alpha(f)\beta(f)\text{-}\lim f_\lambda(x)$, when

(i) $\int_0^\infty |g(x)\psi(x)|\, dx < \infty$, for every $g(x)$ in $\beta(f)$, and

(ii) $\lim_{\lambda \to \infty} \int_0^\infty f_\lambda(x) g(x) dx = \int_0^\infty \psi(x) g(x)\, dx$, for every $g(x)$ in $\beta(f)$

Different $\alpha(f)\beta(f)$-limits of $f_\lambda(x)$ can differ only in sets of x of measure zero.

When $\beta(f) = \alpha^*(f)$, $\psi(x)$ is called a p-limit of $f_\lambda(x)$ in $\alpha(f)$, and we write $\psi(x) = \alpha(f)$ - $\lim f_\lambda(x)$ [cf. I.M., 287].

It follows from the definition that every $\alpha(f)\beta(f)$-limit belongs to $\beta^*(f)$.

When we say that $\psi(x)$ is the $\alpha(f)\beta(f)$-limit of $f_\lambda(x)$, we mean that $\psi(x)$ is an $\alpha(f)\beta(f)$-limit of $f_\lambda(x)$ and every function equal to $\psi(x)$ for almost all $x \geq 0$ is an $\alpha(f)\beta(f)$-limit of $f_\lambda(x)$.

(a) If, with a definition of convergence and limit, every family $f_\lambda(x)$ in $\alpha(f)$, which has a defined limit and also a λ-limit is such that these two limits are equal for almost all $x \geq 0$, then **$\alpha(f)$ is said to be regular** under the defined convergence.

(b) If, with a definition of convergence and limit, every family $f_\lambda(x)$ in $\alpha(f)$, which has a defined limit and also a λ-limit, is such that the λ-limit is in $\alpha(f)$, then $\alpha(f)$ is said to be convergence-closed under that definition of convergence.

(c) If with a definition of convergence and limit, every family $f_\lambda(x)$ in $\alpha(f)$, which has a defined limit is such that this limit is in $\alpha(f)$, then $\alpha(f)$ is said to be limit-closed under the defined convergence [cf. I.M., 290].

2. Some Theorem on projective limit :

Theorem (2.I) : $\alpha^*(f)$ is limit-closed under $\alpha^*(f)$, $\alpha(f)$-convergence.

Proof : From the definition of projective limit it follows that all $\alpha(f)\beta(f)$-limits are in $\beta^*(f)$. Therefore, under $\alpha^*(f)\alpha(f)$-convergence, the projective limit belongs to $\alpha^*(f)$ and so the result follows.

Theorem (2.II) : If $\alpha(f)$ is perfect, it is limit-closed under $\alpha(f)$-convergence.

Proof : Under $\alpha(f)$-convergence, (i.e., $\alpha(f)\,\alpha^*(f)$-convergence), the $\alpha(f)$-limit belong to $\alpha^{**}(f)$; and if $\alpha(f)$ is perfect, $\alpha^{**}(f) = \alpha(f)$.

Hence the result follows.

Theorem (2.III) : When $\alpha^*(f) \geq \beta(f)$, and $\beta(f) \leq \sigma_1(f)$, then every λ-cgt family $f_\lambda(x)$ in $\alpha(f)$ is a $\alpha(f)\beta(f)$-cgt and the λ-limit of $f_\lambda(x)$ is an $\alpha(f)\beta(f)$-limit of $f_\lambda(x)$.

Proof : If $f_\lambda(x)$ is λ-cgt, then to every $\varepsilon > 0$ there corresponds a positive number $N(\varepsilon)$, independent of x, such that, for almost all $x \geq 0$, $|f_\lambda(x) - f_{\lambda'}(x)| \leq \varepsilon$, for all $\lambda, \lambda' \geq N(\varepsilon)$.

Therefore, if $g(x)$ is any function in $\beta(f)$, then

$$\left| \int_0^\infty g(x)\{f_\lambda(x) - f_{\lambda'}(x)\}\,dx \right| \leq \int_0^\infty |g(x)|\,|f_\lambda(x) - f_{\lambda'}(x)|\,dx$$

$$\leq \varepsilon K(g), \quad (\text{since } \beta(f) \leq \sigma_1(f)).$$

for all $\lambda, \lambda' \geq N(\varepsilon)$ for every $\varepsilon > 0$.

Hence $f_\lambda(x)$ is $\alpha(f)\beta(f)$-cgt.

This proves the first part of the theorem.

Next, let $\lambda\text{-}\lim f_\lambda(x) = \psi(x)$. Then given any $\varepsilon > 0$, there exists a positive number $N(\varepsilon)$, independent of x, such that for almost all $x \geq 0$,

$$|f_\lambda(x) - \psi(x)| \leq \varepsilon \tag{2.1}$$

for all $\lambda \geq N(\varepsilon)$.

Choosing any $\varepsilon > 0$, and any $\lambda \geq N(\varepsilon)$.

We have

$$\int_0^\infty |g(x)\,\psi(x)|\,dx = \int_0^\infty |g(x)\{\psi(x) - f_\lambda(x) + f_\lambda(x)\}|\,dx$$

$$\leq \int_0^\infty |g(x)||\psi(x) - f_\lambda(x)|\,dx + \int_0^\infty |g(x)f_\lambda(x)|\,dx$$

$$\leq \varepsilon K(g) + A(g), \text{ by (2.1)}$$

For every, $g(x)$ in $\beta(x)$, since $\beta(f) \leq \sigma_1(f)$ and $\alpha^*(f) \geq \beta(f)$, where K and A are constants depending on g.

Hence

$$\int_0^\infty |g(x)\psi(x)|\,dx < \infty \tag{2.2}$$

for every $g(x)$ in $\beta(f)$.

Again by (2.1),

$$\left| \int_0^\infty f_\lambda(x)g(x)dx - \int_0^\infty \psi(x)\,g(x)dx \right|$$

$$\leq \int_0^\infty |g(x)|\,|f_\lambda(x) - \psi(x)|\,dx \leq \varepsilon\, K(g)$$

for all $\lambda \geq N(\varepsilon)$ for every $\varepsilon > 0$ and for every $g(x)$ in $\beta(f)$.

Therefore,

$$\lim_{\lambda\to\infty} \int_0^\infty f_\lambda(x)\,g(x)dx = \int_0^\infty \psi(x)g(x)\,dx \tag{2.3}$$

for every $g(x)$ in $\beta(f)$.

It follows from (2.2) and (2.3) that $\psi(x)$ is an $\alpha(f)\beta(f)$-limit of $f_\lambda(x)$.

This establishes the second part of the theorem.

We observe from theorem (2.III) that, when $\beta(f) \leq \sigma_1(f)$, $\alpha(f)$ is regular under $\alpha(f)\beta(f)$-convergence.

It for a fixed $\lambda > 0$,

$$f_\lambda(x) = \begin{cases} \psi(x), & \text{for } 0 \le x \le t \\ 0, & \text{for } x > t \end{cases}$$

Then $f_\lambda(x)$ is called a section of $\psi(x)$.

Thus for $\lambda > 0$, we get sections $f_\lambda(x)$ of any given function $\psi(x)$.

Now, we proceed to examine sections of all functions belonging to a given function space. It $\alpha(f)$ is a function space, sections of functions in $\alpha(f)$ may belong to $\alpha(f)$, or may not belong to $\alpha(f)$, as for example, sections of functions in $\sigma_\infty(f)$, $\sigma_p(f)$, where $1 \le p \le \infty$, $\Gamma(f)$, $\varphi(f)$ and $V(f)$ belong respectively to these spaces where as sections of functions in $P(f)$, the space of essentially bounded periodic functions defined in $[0, \infty)$, do not belong to $P(f)$, also the sections of constant functions do not belong to the space of constant functions.

Theorem (2.IV) : If $\alpha(f)$ is such that sections of all functions in $\alpha(f)$ belong to $\alpha(f)$, then every function in $\alpha(f)$ is an $\alpha(f)$-limit of its sections.

Proof : Let $\psi(x) \in \alpha(f)$, and

$$f_\lambda(x) = \begin{cases} \psi(x), & \text{for } 0 \le x \le \lambda \\ 0, & \text{for } x > \lambda \end{cases}$$

Then, by the hypothesis, $f_\lambda(x) \in \alpha(f)$.

Now

$$\int_0^\infty |\psi(x)\, g(x)|\, dx < \infty \tag{2.4}$$

for every $g(x)$ in $\alpha^*(f)$.

Also,

$$\lim_{\lambda\to\infty} \int_0^\infty f_\lambda(x)\, g(x)\, dx = \lim_{\lambda\to\infty} \int_0^\infty \psi(x) g(x)\, dx$$

$$= \int_0^\infty \psi(x)\, g(x)\, dx \tag{2.5}$$

for every $g(x)$ in $\alpha^*(f)$.

Hence, by (2.4) and (2.5)

$$\psi(x) = \alpha(f) - \lim f_{\lambda}(x).$$

And the result follows.

References :

1. Allen, H.S. : "Projective convergence and limit in sequence spaces", P.L.M.S. (2), 48, (1944).
2. Allen, H.S. : "Transformation of sequence spaces", Journal London Math Soc: 31, (1956).
3. Cooke, R.G. : "Infinite matrices and sequence spaces", Macmillan, London, 1950.
4. Cooke, R.G. : "Linear operators", Macmillan, London, 1953.
5. Edward, R.E. : "Functional Analysis" Holt, Rinehart and Winston, Inc., New York, 1965.
6. Goffman, C. and Pedrick, G. : "First Course in Functional Analysis", Prentice Hall, India, New Delhi (1974).
7. Kamthan, P.K. and Gupta, M. : "Sequence Space and Series", Marcel Dekker, New York (1981).