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Existence, Uniqueness and Continuous Dependence for Deformable Derivative Initial Value Problems

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Abstract:

This paper establishes a comprehensive theoretical framework for initial-value problems involving deformable derivatives. Local existence and uniqueness of solutions are proved via the Banach fixed-point theorem under Lipschitz continuity assumptions. These results are extended to global existence under linear growth conditions, continuous dependence on initial conditions and parameters is rigorously demonstrated. Furthermore, the theory is generalized to systems of equations. Collectively, these findings position the deformable derivative as a continuous bridge between integer-order and fractional-order calculus, offering a mathematically tractable and flexible tool for modeling complex physical systems with intermediate dynamics and memory effects.

Keywords: Deformable derivative, Existence and uniqueness, Continuous dependence, Banach fixed point theorem, Systems of differential equations.

1. Introduction:

The concept of a derivative measuring the instantaneous rate of change of a function lies at the heart of mathematical modeling in science and engineering. For centuries, ordinary derivatives of integer order have been successfully used to describe dynamical systems governed by local, memoryless laws. However, many real-world phenomena such as viscoelastic material response, anomalous diffusion, signal processing with memory, and biological systems with hereditary effects-exhibit behavior that cannot be fully captured by classical calculus. These systems often display non-local or intermediate dynamics, lying somewhere between purely algebraic relationships and differential descriptions.

The need to model such in-between behavior motivated the development of fractional calculus, a branch of mathematics that generalizes differentiation and integration to noninteger orders. The origins of fractional calculus trace back to a 1695 correspondence between G.W. Leibniz and G. F. A. de L'Hospital, in which the possibility of a derivative of order one-half was first discussed. Since then, several definitions of fractional derivatives [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12] have been proposed, including those of Riemann-Liouville, Caputo, Grünwald-Letnikov, and more recently, the conformable derivative [13, 14, 15, 16]. Many existing fractional derivatives suffer from certain drawbacks, including the failure to satisfy classical differentiation rules [17], difficulties in interpretation, or the inability to recover the original function at zero order.

In our earlier work, we introduced a simpler and more intuitive generalization called the deformable derivative[18]. For a parameter $k \in [0, 1]$ and $\rho = 1 - k$, the deformable derivative of a function $y(t)$ is defined as

$$D^k y(t) = \rho y(t) + k y'(t).$$

In essence, the deformable derivative is a convex combination of the function itself and its ordinary first derivative. When $k = 0$, it returns the function; when $k = 1$, it reduces to the classical derivative. For intermediate values of k, it smoothly deforms from the function toward its derivative-hence the name. This definition preserves several desirable properties of ordinary calculus, such as linearity, commutativity, and a simple inversion formula, while avoiding the singularities and non-localities associated with many fractional derivatives.

Due to its simplicity and analytic tractability, the deformable derivative has already found applications in several areas, including viscoelasticity, control theory, deformable Laplace transforms [19], and the solution of certain fractional differential equations. However, much of the existing work has focused on the definition, basic properties, and specific applications. A rigorous mathematical theory for initial value problems (IVPs) involving deformable derivatives has remained largely undeveloped. Without such a theory, fundamental questions remain unanswered: Given an IVP of the form

$$D^k y(t) = f(t, y(t)), \qquad y(t_0) = y_0,$$

does a solution exist? Is it unique? Does it depend continuously on the initial data and parameters? Can solutions be extended to large time intervals? These are not merely theoretical concerns-they are essential for ensuring the reliability and interpretability of any mathematical model that employs deformable derivatives.

In this paper, we address these questions systematically and establish a comprehensive theoretical foundation for deformable derivative IVPs. Our main contributions are as follows:

- We prove local existence and uniqueness of solutions under a Lipschitz condition on f, using the Banach fixed point theorem in a suitable metric space of continuous functions.
- We extend the local result to global existence on arbitrary intervals by imposing a linear growth condition on f and employing an iterative continuation argument supported by Gronwall's inequality.
- We establish continuous dependence on initial conditions and parameters, providing explicit stability bounds that ensure small perturbations lead to proportionally small changes in solutions.
- We generalize the theory to systems of deformable differential equations, proving analogous existence, uniqueness, and continuous dependence.

Our approach adapts classical tools from the theory of ordinary differential equations such as fixed-point methods, integral inequalities, and continuation techniques to the specific structure of the deformable derivative. The results not only provide a solid mathematical basis for future applications but also highlight the role of the deformable derivative as a continuous bridge between integer-order and fractional order calculi.

We believe this work will enhance the credibility and utility of deformable derivatives in modeling complex systems with intermediate dynamics. By answering fundamental questions of well-posedness, we hope to encourage further research, both theoretical and applied, into this flexible and intuitively appealing generalization of the derivative.

2. Preliminaries:

This section summarizes the basic mathematical concepts required for the subsequent analysis. Key results from metric space theory, including the Banach fixed point theorem, are recalled. The definition and main properties of the deformable derivative are also presented to fix notation and ensure completeness.

Definition 2.1 (Complete Metric Space [20])**:** A metric space (X, d) is called complete if every Cauchy sequence in X converges to a point in X.

Definition 2.2 (Contraction Mapping [20])**:** Let (X, d) be a metric space. A mapping $T: X \to X$ is called a contraction if there exists a constant $k \in [0, 1)$ such that:

$$d(Tx, Ty) \leq k \cdot d(x, y) \quad \text{for all } x, y \in X.$$

Theorem 2.3 (Banach Fixed Point Theorem [20])**:** Let (X, d) be a complete metric space and $T: X \to X$ be a contraction mapping. Then T has a unique fixed point $x \in X$ such that $T(x) = x$.

Definition 2.4 (Deformable Derivative [18])**:** For $k \in [0, 1]$, the deformable derivative of a function $y : (a, b) \to \mathbb{R}$ at a point $t \in (a, b)$ is defined as

$$D^k y(t) = \lim_{\varepsilon \to 0} \frac{(1 + \varepsilon(1 - k))\, y(t + \varepsilon k) - y(t)}{\varepsilon}.$$

Properties:

- **Linearity:** For all $a, b \in \mathbb{R}$.

$$D^k(ay + bz)(t) = aD^k y(t) + bD^k z(t).$$

- **Product Rule:** $D^k(yz) = (D^k y)\, z + ky\dfrac{dz}{dt}$.
- **Commutativity:** $D^{k_1}D^{k_2} = D^{k_2}D^{k_1}$.
- **Equivalence:** A function is k-differentiable if and only if it is differentiable.

Theorem 2.5: [18] If y is differentiable at $t \in (a, b)$, then it is k-differentiable at t and

$$D^k y(t) = \rho y(t) + ky'(t),$$

where $k + \rho = 1$.

3. Existence and Uniqueness Theorem for Deformable Derivative:

In this section, existence and uniqueness results for deformable derivative IVPs are established. By rewriting the problem in an integral form, the Banach fixed point theorem is applied under standard Lipschitz assumptions. Global existence is obtained using continuation arguments supported by suitable growth conditions.

Theorem 3.1 (Local Existence and Uniqueness)**:** Consider the IVP:

$$D^k y(t) = f(t, y(t)), \quad y(t_0) = y_0, \tag{1}$$

where:

- $k \in (0, 1]$, $\rho = 1 - k$
- f is continuous on the rectangle $R = \{(t, y) : |t - t_0| \le a, |y - y_0| \le b\}$
- f satisfies the Lipschitz condition in y:

$$|f(t, y_1) - f(t, y_2)| \le L|y_1 - y_2| \quad \text{for all } (t, y_1), (t, y_2) \in R.$$

Then there exists $h > 0$ such that the IVP (1) has a unique solution on the interval $I = [t_0 - h, t_0 + h]$.

Proof: Using the relation $D^k y(t) = \rho y(t) + k y'(t)$, equation (1) becomes:

$$\rho y(t) + k y'(t) = f(t, y(t)).$$

Solving for $y'(t)$:

$$y'(t) = \frac{1}{k} f(t, y(t)) - \frac{\rho}{k}\, y(t).$$

Integrating from t_0 to t:

$$y(t) = y_0 + \int_{t_0}^{t} \left(\frac{1}{k} f(s, y(s)) - \frac{\rho}{k} y(s) \right) ds. \tag{2}$$

Define the operator T on continuous functions:

$$(Ty)(t) = y_0 + \int_{t_0}^{t} \left(\frac{1}{k}\, f(s, y(s)) - \frac{\rho}{k} y(s) \right) ds.$$

Let $I = [t_0 - h, t_0 + h]$ with $h > 0$ to be determined. Define the complete metric space:

$$X = \{ y \in C(I) : \| y - y_0 \|_\infty \le b \},$$

where $\| y \|_\infty = \sup_{t \in I} |y(t)|$ and $C(I)$ is the space of continuous functions on I.

Since f is continuous on compact R, it is bounded: $|f(t, y)| \le M$, for all $(t, y) \in R$. For $y \in X$:

$$|(Ty)(t) - y_0| \le \int_{t_0}^{t} \left| \frac{1}{k} f(s, y(s)) - \frac{\rho}{k} y(s) \right| ds.$$

Since $|y(s)| \le |y_0| + b$, we have:

$$\left| \frac{1}{k} f(s, y(s)) - \frac{\rho}{k} y(s) \right| \le \frac{M}{k} + \frac{|\rho|}{k} (|y_0| + b) = M_F.$$

Thus, $|(Ty)(t) - y_0| \le M_F h$.

Choose $h \le \min\left(a, \dfrac{b}{M_F} \right)$ so that $Ty \in X$.

For $y_1, y_2 \in X$:

$$|(Ty_1)(t) - (Ty_2)(t)| \le \int_{t_0}^{t} \left| \frac{1}{k} [f(s, y_1(s)) - f(s, y_2(s))] - \frac{\rho}{k} [y_1(s) - y_2(s)] \right| ds.$$

Using the Lipschitz condition and triangle inequality:

$$\leq \int_{t_0}^{t} \left(\frac{L}{k} + \frac{|\rho|}{k} \right) |y_1(s) - y_2(s)| \, ds \leq \frac{L + |\rho|}{k} h \, \|y_1 - y_2\|_\infty.$$

Taking supremum:

$$\|Ty_1 - Ty_2\|_\infty \leq Kh \, \|y_1 - y_2\|_\infty, \quad \text{where } K = \frac{L + |\rho|}{k}.$$

Choose $h < \min\left(a, \frac{b}{M_F}, \frac{1}{K}\right)$ so that $Kh < 1$, making T a contraction.

Uniqueness:

Assume for contradiction that there exist two distinct solutions y_1 and y_2 in X to the IVP. Both are fixed points of T:

$$Ty_1 = y_1 \quad \text{and} \quad Ty_2 = y_2.$$

Since T is a contraction:

$$\|y_1 - y_2\|_\infty = \|Ty_1 - Ty_2\|_\infty \leq Kh \, \|y_1 - y_2\|_\infty.$$

This implies:

$$(1 - Kh) \, \|y_1 - y_2\|_\infty \leq 0.$$

But $1 - Kh > 0$ because $Kh < 1$, so $\|y_1 - y_2\|_\infty = 0$.

Therefore, $y_1(t) = y_2(t)$ for all $t \in I$, contradicting the assumption. Hence, the solution is unique on I.

Theorem 3.2 (Global Existence and Uniqueness Theorem): Consider the IVP

$$D^k y(t) = f(t, y(t)), \quad y(t_0) = y_0, \tag{3}$$

on a finite interval $J = [t_0, T]$. Assume that:

1. The function $f(t, y)$ is continuous on $[t_0, T] \times \mathbb{R}$.
2. f satisfies a global Lipschitz condition in y: There exists $L > 0$ such that for all $t \in [t_0, T]$ and all $y_1, y_2 \in \mathbb{R}$,

$$|f(t, y_1) - f(t, y_2)| \leq L |y_1 - y_2|.$$

3. f has at most linear growth in y: There exist constants $A, B \geq 0$ such that

$$|f(t, y)| \leq A|y| + B \text{ for all } t \in [t_0, T], y \in \mathbb{R}.$$

Then, the IVP (1) has a unique solution $y(t)$ defined on the entire interval J.

Proof: The deformable derivative IVP (1) is equivalent to the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} \left(\frac{1}{k} f(s, y(s)) - \frac{\rho}{k} y(s)\right) ds, \quad (4)$$

where $k \in (0, 1]$ and $\rho = 1 - k$.

Define the contraction constant

$$K = \frac{L + |\rho|}{k}.$$

Fix an arbitrary $b > 0$ defining the solution tube around the initial value. Using the linear growth condition, we bound the integrand in (2). For any function $y(s)$ satisfying $|y(s) - y_0| \leq b$ (so $|y(s)| \leq |y_0| + b$), we have:

$$\begin{aligned} \left|\frac{1}{k} f(s, y(s)) - \frac{\rho}{k} y(s)\right| &\leq \frac{1}{k} |f(s, y(s))| + \frac{|\rho|}{k} |y(s)| \\ &\leq \frac{1}{k} (A(|y_0| + b) + B) + \frac{|\rho|}{k} (|y_0| + b) \\ &= \frac{(A + |\rho|)(|y_0| + b) + B}{k}. \end{aligned}$$

Thus, define

$$M_F = \frac{B}{k} + \frac{|\rho| + A}{k} (|y_0| + b).$$

Choose the step size h satisfying

$$h < \min\left(T - t_0, \frac{b}{M_F}, \frac{1}{K}\right).$$

This ensures all conditions of Theorem 3.1 are met for the first local step.

By Theorem 3.1, there exists a unique solution $y_1(t)$ to the IVP (1) on the interval $I_1 = [t_0, t_0 + h]$.

Iterative Continuation:

Define $t_1 = t_0 + h$ and consider the new IVP with initial condition $y(t_1) = y_1(t_1)$. The assumptions (1)-(3) remain valid on $[t_1, T] \times \mathbb{R}$.

To continue the process uniformly, we establish an a priori bound on the solution. Let $y(t)$ be any solution of (1) on some subinterval. From the integral formulation (2) and the linear growth condition:

$$|y(t)| \le |y_0| + \int_{t_0}^{t} \left(\frac{1}{k} |f(s, y(s))| + \frac{|\rho|}{k} |y(s)| \right) ds,$$

$$\le |y_0| + \int_{t_0}^{t} \left(\frac{1}{k} (A|y(s)| + B) + \frac{|\rho|}{k} |y(s)| \right) ds$$

$$= |y_0| + \frac{B}{k}(t - t_0) + \frac{A + |\rho|}{k} \int_{t_0}^{t} |y(s)|\, ds.$$

By Gronwall's inequality, there exists a constant $M > 0$ (depending on $|y_0|$, T, A, B, k, ρ) such that

$$|y(t)| \le M \text{ for all } t \in [t_0, T].$$

Using this uniform bound M, we can choose a uniform step size $h_{\text{uniform}} > 0$ valid for all continuation steps. Specifically, take $b = 1$ (any positive value suffices) and define

$$M_F^{\text{uniform}} = \frac{B}{k} + \frac{|\rho| + A}{k}(M + 1),$$

and choose

$$h_{\text{uniform}} < \min\left(T - t_0, \frac{1}{M_F^{\text{uniform}}}, \frac{1}{K} \right).$$

Now proceed iteratively:

- **Step 1:** Solution $y_1(t)$ on $[t_0, t_0 + h_{\text{uniform}}]$
- **Step 2:** Solution $y_2(t)$ on $[t_0 + h_{\text{uniform}}, t_0 + 2h_{\text{uniform}}]$ with initial condition $y(t_0 + h_{\text{uniform}}) = y_1(t_0 + h_{\text{uniform}})$
- **Step k:** Continue until $t_0 + kh_{\text{uniform}} \ge T$

After a finite number of steps n (specifically, $n = [(T - t_0)/h_{\text{uniform}}]$), we obtain a solution defined on the entire interval $[t_0, T]$ by concatenation:

$$y(t) = y_k(t) \quad \text{for } t \in [t_0 + (k-1)h_{\text{uniform}}, t_0 + kh_{\text{uniform}}], \quad k = 1, \ldots.., n.$$

Global Uniqueness:

Suppose $z(t)$ is another solution to (1) on J. On the first subinterval $I_1 = [t_0, t_0 + h_{\text{uniform}}]$, both $y(t)$ and $z(t)$ satisfy the IVP. By Theorem 3.1 (local uniqueness), $y(t) = z(t)$ on I_1.

In particular, $y(t_0 + h_{\text{uniform}}) = z(t_0 + h_{\text{uniform}})$. On the next subinterval $I_2 = [t_0 + h_{\text{uniform}}, t_0 + 2h_{\text{uniform}}]$, both functions satisfy the IVP with the same initial condition at $t_0 + h_{\text{uniform}}$. Again by local uniqueness, $y(t) = z(t)$ on I_2.

Continuing this argument inductively through all subintervals, we conclude that $y(t) = z(t)$ for all $t \in [t_0, T]$.

Therefore, the solution exists and is unique on the entire interval J.

4. Continuous Dependence on Initial Conditions:

This section studies the sensitivity of solutions of deformable derivative IVPs with respect to variations in initial conditions and parameters. By comparing two solutions corresponding to nearby initial data and using Lipschitz continuity together with Gronwall-type inequalities, explicit bounds are derived. These results confirm that the problem is well-posed in the sense of continuous dependence on initial conditions.

Theorem 4.1 (Continuous Dependence on Initial Conditions): Let $y(t)$ and $z(t)$ be solutions of the deformable derivative IVP

$$D^k y(t) = f(t, y(t)), \qquad y(t_0) = y_0,$$

and

$$D^k z(t) = f(t, z(t)), \qquad z(t_0) = z_0,$$

respectively, on the interval $I = [t_0 - h, t_0 + h]$. Assume that f satisfies the Lipschitz condition in y with constant $L > 0$ on a region containing the graphs of y and z, i.e.

$$|f(t, y_1) - f(t, y_2)| \le L\,|y_1 - y_2|$$

for all (t, y_1), (t, y_2) in that region.

Then, for every $t \in I$,

$$|y(t) - z(t)| \le |y_0 - z_0|\, e^{K|t-t_0|}, \tag{5}$$

where

$$K = \frac{L + |\rho|}{k}, \quad \rho = 1 - k.$$

Proof: From the equivalent integral formulation (4), we have

$$y(t) = y_0 + \int_{t_0}^{t} \left[\frac{1}{k}\, f(s, y(s)) - \frac{\rho}{k}\, y(s)\right] ds,$$

$$z(t) = z_0 + \int_{t_0}^{t} \left[\frac{1}{k}\, f(s, z(s)) - \frac{\rho}{k}\, z(s)\right] ds.$$

Subtracting the two expressions gives

$$y(t) - z(t) = y_0 - z_0 + \int_{t_0}^{t} \left[\frac{1}{k}\,(f(s, y(s)) - f(s, z(s)) - \frac{\rho}{k}\,(y(s) - z(s))\right] ds.$$

Taking absolute values and using the triangle inequality yields

$$|y(t) - z(t)| \le |y_0 - z_0| + \int_{t_0}^{t} \left[\frac{1}{k}\,|f(s, y(s)) - f(s, z(s))| + \frac{|\rho|}{k}\,|y(s) - z(s)|\right] ds.$$

Applying the Lipschitz condition, $|f(s, y(s)) - f(s, z(s))| \le L|y(s) - z(s)|$, we obtain

$$|y(t) - z(t)| \le |y_0 - z_0| + \int_{t_0}^{t} \left(\frac{L}{k} + \frac{|\rho|}{k}\right) |y(s) - z(s)|\, ds.$$

Set $K = (L + |\rho|)/k$. Then the inequality becomes

$$|y(t) - z(t)| \le |y_0 - z_0| + K \int_{t_0}^{t} |y(s) - z(s)|\, ds.$$

This is a Gronwall-type inequality. Applying the integral form of Gronwall's lemma gives precisely

$$|y(t) - z(t)| \le |y_0 - z_0|\, e^{K|t-t_0|},$$

which completes the proof.

Remark 4.2: The constant K depends on the deformable parameter k through $|\rho|/k$ and on the Lipschitz constant L of f. For $k = 1$ (classical derivative) we have $\rho = 0$ and $K = L$, recovering the standard ODE estimate. For $k < 1$ the term $|\rho|/k$ adds to the exponential growth rate, reflecting the influence of the "memory" term $\rho y(t)$ in the deformable derivative.

Corollary 4.3 (Parameter Dependence): Let $f(t, y, \lambda)$ depend continuously on a parameter λ and satisfy a uniform Lipschitz condition in y,

$$|f(t, y_1, \lambda) - f(t, y_2, \lambda)| \leq L\,|y_1 - y_2|$$

Let $y(t, \lambda)$ denote the solution of

$$D^k y = f(t, y, \lambda), \quad y(t_0) = y_0(\lambda),$$

where $y_0(\lambda)$ is continuous in λ. Then $y(t, \lambda)$ depends continuously on λ.

Proof: Take two parameter values λ_1, λ_2 and apply Theorem 4.1 to the corresponding solutions $y(t, \lambda_1)$ and $y(t, \lambda_2)$. The difference of the initial values is controlled by the continuity of $y_0(\lambda)$, and the Lipschitz condition on f is uniform in λ. Gronwall's inequality then yields

$$|y(t, \lambda_1) - y(t, \lambda_2)| \leq \Big(|y_0(\lambda_1) - y_0(\lambda_2)| + \varepsilon(\lambda_1, \lambda_2)\Big)\, e^{K|t-t_0|},$$

where $\varepsilon(\lambda_1, \lambda_2) \to 0$ as $\lambda_1 \to \lambda_2$ by the continuity of f in λ. Hence $y(t, \lambda)$ is continuous in λ.

Corollary 4.4 (Uniform Dependence Bound): Under the hypotheses of Theorem 4.1, if $|y_0 - z_0| < \delta$, then

$$\| y - z \|_{L^\infty(I)} \leq \delta\, e^{Kh}.$$

Thus small perturbations of the initial datum produce proportionally small changes in the solution, uniformly on the interval I.

Proof: From Theorem 4.1, for any $t \in I = [t_0 - h, t_0 + h]$ we have

$$|y(t) - z(t)| \leq |y_0 - z_0|\, e^{K|t-t_0|} \leq \delta\, e^{Kh},$$

since $|t - t_0| \leq h$. Taking the supremum over $t \in I$ gives the stated uniform bound.

The estimate (5) shows that solutions depend continuously on the initial conditions, with a stability constant K that grows when k decreases. This behaviour is expected because the deformable derivative $D^k y = \rho y + ky'$ contains the term ρy (with $\rho = 1 - k$), which acts as a self-feedback or memory-like term. As $k \to 0$ this term dominates and can amplify differences between solutions, leading to a larger exponential factor in the stability bound. Nevertheless, the well-posedness of the problem is guaranteed as long as the Lipschitz condition holds.

5. Systems of Deformable Derivative Equations:

The theory developed for scalar deformable derivative IVPs extends naturally to systems. This section presents the generalization of existence, uniqueness, and continuous dependence results to systems of deformable differential equations.

Consider a vector-valued function

$$y(t) = (y_1(t), y_2(t), \ldots, y_n(t))^T : [t_0, T] \to \mathbb{R}^n,$$

and a continuous vector field

$$f(t, y) = (f_1(t, y), f_2(t, y), \ldots, f_n(t, y))^T : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n.$$

The deformable derivative operator is applied componentwise:

$$D^k y(t) = (D^k y_1(t), D^k y_2(t), \ldots, D^k y_n(t))^T,$$

where for each component,

$$D^k y_i(t) = \rho y_i(t) + k y_i'(t), \quad \rho = 1 - k.$$

IVP for Systems:

We study the system

$$D^k y(t) = f(t, y(t)), \quad y(t_0) = y_0, \tag{6}$$

where $y_0 \in \mathbb{R}^n$ is given.

The system (6) is equivalent to the vector integral equation

$$y(t) = y_0 + \int_{t_0}^{t} \left[\frac{1}{k} f(s, y(s)) - \frac{\rho}{k} y(s) \right] ds. \tag{7}$$

Let $I = [t_0 - h, t_0 + h]$ be a closed interval. Denote by $C(I, \mathbb{R}^n)$ the space of continuous vector-valued functions on I with values in $\mathbb{R}^n$. Equip this space with the supremum norm

$$\|y\|_\infty = \sup_{t \in I} \|y(t)\|,$$

where $\|\cdot\|$ is any norm on $\mathbb{R}^n$ (typically the Euclidean norm or the maximum norm).

***Existence and Uniqueness for Systems*:**

Theorem 5.1 (Local Existence and Uniqueness for Systems): Consider the system (6) with $k \in (0, 1]$. Let

$$R = \{(t, y) : |t - t_0| \le a,\ \|y - y_0\| \le b\}$$

be a closed rectangle in $[t_0, T] \times \mathbb{R}^n$. Assume that:

1. f is continuous on R.
2. f satisfies a Lipschitz condition in y uniformly in t: there exists $L > 0$ such that

   $$\|f(t, y_1) - f(t, y_2)\| \le L\,\|y_1 - y_2\|$$

   for all $(t, y_1), (t, y_2) \in R$.

Then there exists $h > 0$ such that the system (6) has a unique solution $y(t)$ on the interval $I = [t_0 - h, t_0 + h]$.

Proof: Define the operator $T : C(I, \mathbb{R}^n) \to C(I, \mathbb{R}^n)$ by

$$(Ty)(t) = y_0 + \int_{t_0}^{t} \left[\frac{1}{k} f(s, y(s)) - \frac{\rho}{k} y(s) \right] ds.$$

Let

$$X = \{y \in C(I, \mathbb{R}^n) : \|y - y_0\|_\infty \le b\}.$$

Since f is continuous on the compact set R, it is bounded: $\|f(t, y)\| \le M$ for all $(t, y) \in R$.

For $y \in X$,

$$\|(Ty)(t) - y_0\| \le \int_{t_0}^{t} \left\| \frac{1}{k} f(s, y(s)) - \frac{\rho}{k} y(s) \right\| ds$$

$$\le \int_{t_0}^{t} \left(\frac{1}{k} \|f(s, y(s))\| + \frac{|\rho|}{k} \|y(s)\| \right) ds$$

$$\le \left(\frac{M}{k} + \frac{|\rho|}{k} (\|y_0\| + b) \right) |t - t_0|.$$

Set

$$M_F = \frac{M}{k} + \frac{|\rho|}{k} (\|y_0\| + b).$$

Choose $h \le \min\left(a, \frac{b}{M_F}\right)$; then $\|Ty - y_0\|_\infty \le b$, so T maps X into itself.

Now, for $y_1, y_2 \in X$,

$$\|(Ty_1)(t) - (Ty_2)(t)\| \le \int_{t_0}^{t} \left\| \frac{1}{k} [f(s, y_1(s)) - f(s, y_2(s))] - \frac{\rho}{k} [y_1(s) - y_2(s)] \right\| ds$$

$$\le \int_{t_0}^{t} \left(\frac{L}{k} + \frac{|\rho|}{k} \right) \|y_1(s) - y_2(s)\| \, ds$$

$$\le \frac{L + |\rho|}{k} h \, \|y_1 - y_2\|_\infty.$$

Thus

$$\|Ty_1 - Ty_2\|_\infty \le Kh \, \|y_1 - y_2\|_\infty,$$

where $K = \frac{L + |\rho|}{k}$.

Choosing $h < \min\left(a, \frac{b}{M_F}, \frac{1}{K}\right)$ ensures $Kh < 1$, making T a contraction on the complete metric space X. By the Banach fixed point theorem, T has a unique fixed point, which is the unique solution of (6) on I.

***Global Existence for Systems*:**

Theorem 5.2 (Global Existence and Uniqueness for Systems): Assume that:

1. $f : [t_0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

2. f satisfies a global Lipschitz condition in y: there exists $L > 0$ such that

$$\| f(t, y_1) - f(t, y_2) \| \le L \| y_1 - y_2 \|$$

for all $t \in [t_0, T]$ and all $y_1, y_2 \in \mathbb{R}^n$.

3. f has at most linear growth: there exist constants $A, B \ge 0$ such that

$$\| f(t, y) \| \le A \| y \| + B \text{ for all } t \in [t_0, T] \text{ and } y \in \mathbb{R}^n.$$

Then the system (6) has a unique solution $y(t)$ defined on the entire interval $[t_0, T]$.

Proof: The proof follows the same strategy as the scalar global existence result established in Section 3. First, an a priori bound is obtained. This uniform bound allows the choice of a uniform step size

$$h_{\text{uni}} < \min \left\{ T - t_0, \frac{1}{M_F^{\text{uni}}}, \frac{1}{K} \right\},$$

where

$$M_F^{\text{uni}} = \frac{B}{k} + \frac{|\rho| + A}{k}(M+1), \ K = \frac{L + |\rho|}{k}.$$

Iterative application of Theorem 5.1 with step size h_{uni} covers the interval $[t_0, T]$ in finitely many steps. Uniqueness on each subinterval implies global uniqueness.

Continuous Dependence for Systems:

Theorem 5.3 (Continuous Dependence for Systems): Let $y(t)$ and $z(t)$ be solutions of

$$D^k y(t) = f(t, y(t)), \quad y(t_0) = y_0,$$

and

$$D^k z(t) = f(t, z(t)), \quad z(t_0) = z_0,$$

on an interval $I = [t_0 - h, t_0 + h]$. Assume f satisfies the Lipschitz condition

$$\| f(t, y_1) - f(t, y_2) \| \le L \| y_1 - y_2 \|,$$

on a region containing the graphs of y and z. Then, for all $t \in I$,

$$\| y(t) - z(t) \| \leq \| y_0 - z_0 \| \, e^{K|t-t_0|},$$

where $K = \dfrac{L + |\rho|}{k}$.

Proof: The proof is identical to the scalar case (Theorem 4.1), with absolute values replaced by vector norms. The integral inequality

$$\| y(t) - z(t) \| \leq \| y_0 - z_0 \| + K \int_{t_0}^{t} \| y(s) - z(s) \| \, ds$$

follows directly from the Lipschitz condition and the triangle inequality. Application of Gronwall's inequality (valid for vector-valued functions) yields the exponential bound.

6. Conclusion:

This paper has established a comprehensive and rigorous theoretical framework for initial value problems involving deformable derivatives. We began by proving the local existence and uniqueness of solutions under a Lipschitz condition on the function f using the Banach fixed-point theorem in an appropriate metric space of continuous functions. This foundational result was then extended to global existence on arbitrary time intervals by imposing a linear growth condition on f and employing an iterative continuation argument supported by Gronwall's inequality. Furthermore, we demonstrated the continuous dependence of solutions on initial conditions and parameters, providing explicit exponential dependence bounds that ensure robustness against small perturbations. Finally, we generalized the entire theory to systems of deformable differential equations, establishing analogous well-posedness results for vector-valued problems. The deformable derivative, defined as $D^k y = \rho y + k y'$ with $\rho = 1 - k$, serves as a mathematically tractable and physically intuitive operator that continuously interpolates between a function and its classical derivative. This work not only fills a significant theoretical gap in the existing literature but also provides a solid mathematical foundation for future applications of deformable derivatives in modeling complex systems with intermediate dynamics and memory effects, such as in viscoelasticity, control theory, signal processing, and anomalous diffusion. We anticipate that these results will encourage further research, both theoretical and applied, into this versatile and promising generalization of classical calculus.

Future directions may include the study of boundary-value problems, stability analysis of deformable derivative systems, numerical methods tailored to the deformable structure, and applications in data-driven modeling where the parameter k can be learned from experimental observations.

Key Contributions:

- A complete local and global existence and uniqueness theory for deformable derivative IVPs.
- Explicit stability bounds demonstrating continuous dependence on data.
- Extension to systems of equations, enabling multi-dimensional modeling.
- Clarification of the role of the parameter k as a continuous bridge between integer-order and fractional-order calculus.

By answering fundamental questions of well-posedness, this work enhances the credibility and utility of deformable derivatives as a flexible tool for describing phenomena that lie between purely algebraic relationships and classical differential descriptions.

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