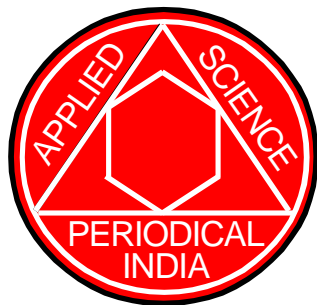


ISSN 0972-5504

VOLUME - XXIII, No. 4, NOVEMBER 2021

APPLIED SCIENCE PERIODICAL

Since 1999



*A Refereed and Peer-Reviewed
Quarterly Periodical Devoted To
Mathematical and Natural Sciences*

Applied Science Periodical

ISSN 0972-5504

Volume - XXIII, No. 4, November 2021

Refereed and Peer-Reviewed Quarterly Periodical

Journal website : www.internationaljournalsiwan.com

Some Transforms of the Fractional Integral Operators Involving General Polynomial, Multivariable Mittag-Leffler Function and Modified I-Function of Two Variables

by **Harendra Singh,**

Department of Mathematics,

M.M.H.College, Ghaziabad - 201001, India

E-mail : sharendra9@yahoo.com

&

Prvindra Kumar,

Department of Mathematics,

D.J.College, Baraut - 250611, India

E-mail : prvindradjc@gmail.com

Abstract :

In literature, a lot of remarkable pathway type fractional integral operators whose integrand include various special functions have been given. In this paper, first we establish two theorems involving the composition of Srivastava's polynomial, multivariable Mittag-Leffler function and modified I-function of two variables with the pathway type fractional integral operators. In the next section we shall obtain some transforms like Beta, Laplace, Hankel, Verma and K-transforms of the composition of Srivastava's polynomial, multivariable Mittag-Leffler function and modified I-function of two variables with the pathway type fractional integral operators.

Keywords and phrases : Pathway type fractional integral operators, Srivastava's polynomial, multivariable Mittag-Leffler function, modified I -function, Beta transform, Laplace transform, Verma Transform, Hankel transform, K -transform.

Mathematics Subject Classification (2010) : Primary: 26A33, 33C60, 33C99; Secondary: 44A20, 33E12, 05C38.

1. Introduction and preliminaries :

In 1979, Prasad and Prasad [8] introduced modified H -function of two variables and in 2012, Shantha Kumari et al. [3] defined I -function of two variables. In this paper, we are defining modified I -function of two variables, which is the generalization of both the modified H -function of two variables of Prasad and Prasad and I -function of two variables Shantha Kumari, in the following manner:

$$\begin{aligned}
 I[z_1, z_r] &= I_{p, q: p_1, q_1: p_2, q_2: p_3, q_3}^{m, n: m_1, n_1: m_2, n_2: m_3, n_3} \\
 &\left[\begin{array}{l} z_1 | (a_j; \alpha_j, A_j; \xi_j)_{1, p} : (c_j; \gamma_j, C_j; \xi'_j)_{1, p_1} : (e_j, E_j; U_j)_{1, p_2} ; (g_j, G_j; P_j)_{1, p_3} \\ z_2 | (b_j; \beta_j, B_j; \eta_j)_{1, q} : (d_j; \delta_j, D_j; \eta'_j)_{1, q_1} : (f_j, F_j; V_j)_{1, q_2} ; (h_j, H_j; Q_j)_{1, q_3} \end{array} \right] \\
 &= \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \psi(s_1, s_2) \theta_1(s_1) \theta_2(s_2) z_1^{s_1} z_2^{s_2} ds_1 ds_2 \tag{1}
 \end{aligned}$$

where

$$\begin{aligned}
 \psi(s_1, s_2) &= \frac{\prod_{j=1}^m \Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2) \prod_{j=1}^n \Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 + A_j s_2)}{\prod_{j=1}^q \Gamma^{\eta_j}(1 - b_j + \beta_j s_1 + B_j s_2) \prod_{j=n+1}^p \Gamma^{\xi_j}(a_j - \alpha_j s_1 - A_j s_2)} \\
 &\times \frac{\prod_{j=1}^{m_1} \Gamma^{\eta'_j}(d_j - \delta_j s_1 + D_j s_2) \prod_{j=1}^{n_1} \Gamma^{\xi'_j}(1 - c_j + \gamma_j s_1 - C_j s_2)}{\prod_{j=1}^{q_1} \Gamma^{\eta'_j}(1 - d_j + \delta_j s_1 - D_j s_2) \prod_{j=n+1}^{p_1} \Gamma^{\xi'_j}(c_j - \gamma_j s_1 + C_j s_2)} \tag{2}
 \end{aligned}$$

$$\theta_1(s_1) = \frac{\prod_{j=1}^{m_2} \Gamma^{U_j}(f_j - F_j s_1) \prod_{j=1}^{n_2} \Gamma^{V_j}(1 - e_j + E_j s_1)}{\prod_{j=m_2+1}^{q_2} \Gamma^{U_j}(1 - f_j + F_j s_1) \prod_{j=n_2+1}^{p_2} \Gamma^{V_j}(e_j - E_j s_1)} \quad (3)$$

$$\theta_2(s_2) = \frac{\prod_{j=1}^{m_3} \Gamma^{P_j}(h_j - H_j s_2) \prod_{j=1}^{n_3} \Gamma^{Q_j}(1 - g_j + G_j s_2)}{\prod_{j=m_3+1}^{q_3} \Gamma^{P_j}(1 - h_j + H_j s_2) \prod_{j=n_3+1}^{p_3} \Gamma^{Q_j}(g_j - G_j s_2)} \quad (4)$$

Here, the variables z_1 and z_2 are non-zero real or complex numbers and an empty product is interpreted as unity. $m, n, m_1, n_1, m_2, n_2, m_3, n_3, p, q, p_1, q_1, p_2, q_2, p_3, q_3$ are all non-negative integers such that $0 \leq n \leq p, 0 \leq m \leq q, 0 \leq n_1 \leq p_1, 0 \leq m_1 \leq q_1, 0 \leq n_2 \leq p_2, 0 \leq m_2 \leq q_2, 0 \leq n_3 \leq p_3, 0 \leq m_3 \leq q_3$ and $\alpha_j, \beta_j, \gamma_j, \delta_j, A_j, B_j, C_j, D_j, E_j, F_j, G_j, H_j, \xi_j, \eta_j, \xi'_j, \eta'_j, U_j, V_j, P_j, Q_j$ are all positive real numbers. $a_j, b_j, c_j, d_j, e_j, f_j, g_j, h_j$, are all complex numbers. The integration path L_1 in the complex s_1 plane runs from $\sigma_1 - i\infty$ to $\sigma_1 + i\infty$ such that all the poles of $\Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2)$ for $j = 1, \dots, m$, $\Gamma^{\eta'_j}(d_j - \delta_j s_1 + D_j s_2)$ for $j = 1, \dots, m_1$ and $\Gamma^{U_j}(f_j - F_j s_1)$ for $j = 1, \dots, m_2$ lie to the right of L_1 while all the poles of $\Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 + A_j s_2)$ for $j = 1, \dots, n$, $\Gamma^{\xi'_j}(1 - c_j + \gamma_j s_1 - C_j s_2)$ for $j = 1, \dots, n_1$ and $\Gamma^{V_j}(1 - e_j + E_j s_1)$ for $j = 1, \dots, n_2$ lie to the left of L_1 . The integration path L_2 in the complex s_2 plane runs from $\sigma_2 - i\infty$ to $\sigma_2 + i\infty$ such that all the poles of $\Gamma^{\eta_j}(b_j - \beta_j s_1 - B_j s_2)$ for $j = 1, \dots, m$, $\Gamma^{\xi'_j}(1 - c_j + \gamma_j s_1 - C_j s_2)$ for $j = 1, \dots, n_1$ and $\Gamma^{P_j}(h_j - H_j s_2)$ for $j = 1, \dots, m_3$ lie to the right of L_2 while all the poles of $\Gamma^{\xi_j}(1 - a_j + \alpha_j s_1 + A_j s_2)$ for $j = 1, \dots, n$, $\Gamma^{\eta'_j}(d_j - \delta_j s_1 + D_j s_2)$ for $j = 1, \dots, m_1$ and $\Gamma^{Q_j}(1 - g_j + G_j s_2)$ for $j = 1, \dots, n_3$ lie to the left of L_2 . When the exponents of various gamma functions in equations (2), (3) and (4) are not integers, the poles of gamma functions in numerator are converted to branch points and branch cuts can be chosen that the paths of integration can be distorted for each of contours as long as that there is no coincidence of poles.

Using the results of Braaksma [5] and Rathie [9], it is easy to show that the function $I[z_1, z_2]$ defined by (1) is an analytic function of z_1 and z_2 if

$$V_1 = \sum_{j=1}^p \xi_j \alpha_j + \sum_{j=1}^{p_1} \xi'_j \gamma_j + \sum_{j=1}^{p_2} U_j E_j - \sum_{j=1}^q \eta_j \beta_j - \sum_{j=1}^{q_1} \eta'_j \delta_j - \sum_{j=1}^{q_2} V_j F_j < 0 \quad (5)$$

$$V_2 = \sum_{j=1}^p \xi_j A_j + \sum_{j=1}^{q_1} \eta'_j D_j + \sum_{j=1}^{p_3} P_j G_j - \sum_{j=1}^q \eta_j B_j - \sum_{j=1}^{p_1} \xi'_j C_j - \sum_{j=1}^{q_3} Q_j H_j < 0 \quad (6)$$

Similarly following Shantha Kumari et al. [3] and Prasad and Prasad [8], it is easy to show that the double integral defined in (1) converges absolutely if

$$|\arg z_1| < \frac{1}{2} \pi \Omega_1 \text{ and } |\arg z_2| < \frac{1}{2} \pi \Omega_2 \text{ where}$$

$$\begin{aligned} \Omega_1 = & \sum_{j=1}^n \xi_j \alpha_j - \sum_{j=n+1}^p \xi_j \alpha_j + \sum_{j=1}^m \eta_j \beta_j - \sum_{j=m+1}^q \eta_j \beta_j + \sum_{j=1}^{n_1} \xi'_j \gamma_j - \sum_{j=n_1+1}^{p_1} \xi'_j \gamma_j + \sum_{j=1}^{m_1} \eta'_j \delta_j \\ & - \sum_{j=m_1+1}^{q_1} \eta'_j \delta_j + \sum_{j=1}^{n_2} U_j E_j - \sum_{j=n_2+1}^{p_2} U_j E_j + \sum_{j=1}^{m_2} V_j F_j - \sum_{j=m_2+1}^{q_2} V_j F_j > 0 \end{aligned} \quad (7)$$

and

$$\begin{aligned} \Omega_2 = & \sum_{j=1}^n \xi_j A_j - \sum_{j=n+1}^p \xi_j A_j + \sum_{j=1}^m \eta_j B_j - \sum_{j=m+1}^q \eta_j B_j + \sum_{j=1}^{n_1} \xi'_j C_j - \sum_{j=n_1+1}^{p_1} \xi'_j C_j + \sum_{j=1}^{m_2} \eta'_j D_j \\ & - \sum_{j=m_1+1}^{q_2} \eta'_j D_j + \sum_{j=1}^{n_3} P_j G_j - \sum_{j=n_3+1}^{p_3} P_j G_j + \sum_{j=1}^{m_3} Q_j H_j - \sum_{j=m_3+1}^{q_3} Q_j H_j > 0 \end{aligned} \quad (8)$$

Now the asymptotic behavior may be establish in the following convenient form, see Braaksma [1].

$$I(z_1, z_2) = 0 \quad (|z_1|^{\alpha_1}, |z_2|^{\alpha_2}), \max (|z_1|^{\alpha_1}, |z_2|^{\alpha_2}) \rightarrow 0$$

$$I(z_1, z_2) = 0 \quad (|z_1|^{\beta_1}, |z_2|^{\beta_2}), \min (|z_1|^{\beta_1}, |z_2|^{\beta_2}) \rightarrow \infty, \text{ where}$$

$$\alpha_1 = \min_{1 \leq j \leq m_2} \Re \left[V_j \left(\frac{f_j}{F_j} \right) \right] \text{ and } \alpha_2 = \min_{1 \leq j \leq m_3} \Re \left[Q_j \left(\frac{h_j}{H_j} \right) \right]$$

$$\beta_1 = \max_{1 \leq j \leq n_2} \Re \left[U_j \left(\frac{e_j - 1}{E_j} \right) \right] \text{ and } \beta_2 = \max_{1 \leq j \leq n_3} \Re \left[P_j \left(\frac{g_j - 1}{G_j} \right) \right]$$

For simplicity, we shall use the following notations;

$$U = m_1, n_1 : m_2, n_2 : m_3, n_3 \tag{9}$$

$$V = p_1, q_1 : p_2, q_2 : p_3, q_3 \tag{10}$$

$$A_1 = (a_j; \alpha_j, A_j; \xi_j)_{1,p} \tag{11}$$

$$A_2 = (c_j; \gamma_j, C_j; \xi'_j)_{1,p_1} \tag{12}$$

$$A_3 = (e_j; E_j; U_j)_{1,p_2} \tag{13}$$

$$A_4 = (g_j; G_j; P_j)_{1,p_3} \tag{14}$$

$$B_1 = (b_j; \beta_j, B_j; \eta_j)_{1,q} \tag{15}$$

$$B_2 = (d_j; \delta_j, D_j; \eta'_j)_{1,q_1} \tag{16}$$

$$B_3 = (f_j; F_j; V_j)_{1,q_3} \tag{17}$$

$$B_4 = (h_j; H_j; Q_j)_{1,q_3} \tag{18}$$

In 1903, Mittag-Leffler [5] introduced the function $E_\alpha(y)$ in the following manner:

$$E_\alpha(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + 1)}, \text{ where } \alpha, y \in C, \Re(\alpha) > 0 \tag{19}$$

In 1905, Wiman [13] generalized the function $E_\alpha(y)$ and gave the function $E_{\alpha,\beta}(y)$ in the following manner:

$$E_{\alpha,\beta}(y) = \sum_{k=0}^{\infty} \frac{y^k}{\Gamma(\alpha k + \beta)}, \text{ where } \alpha, \beta, y \in C, \Re(\alpha) > 0, \Re(\beta) > 0 \tag{20}$$

In 1971, Prabhakar [7] generalized the function $E_{\alpha,\beta}(y)$ and gave the function $E_{\alpha,\beta}^\lambda(y)$ in the following manner:

$$E_{\alpha,\beta}^\lambda(y) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(\alpha k + \beta)} \frac{y^k}{k!}, \tag{21}$$

where $\alpha, \beta, \lambda, y \in C, \Re(\alpha) > 0, \Re(\beta) > 0, \Re(\lambda) > 0$

Saxena et al. [10] gave the multivariable analogue of multivariable Mittag-Leffler function in the following manner:

$$E_{\mu_i, \nu_1}^{\lambda_i}(y_1, \dots, y_m) = E_{\mu_1, \dots, \mu_m, \nu_1}^{\lambda_1, \dots, \lambda_m}(y_1, \dots, y_m) = \sum_{k_1, \dots, k_m=0}^{\infty} \frac{(\lambda_1)_{k_1} \dots (\lambda_m)_{k_m}}{\Gamma\left(\nu_1 + \sum_{i=1}^m \mu_i k_i\right)} \frac{y_1^{k_1}}{k_1!} \dots \frac{y_m^{k_m}}{k_m!} \quad (22)$$

where $\nu_1, \lambda_i, \mu_i \in C, \Re(\mu_j) > 0, \forall i = 1, 2 \dots m$.

For the sake of convenience, let $F_{\mu_i, c}^{\lambda_i} = \frac{(\lambda_i)_{k_1} \dots (\lambda_i)_{k_l}}{\Gamma\left(c + \sum_{i=1}^l \mu_i k_i\right)}$ (23)

Srivastava [13] introduced the general class of polynomials in the following manner:

$$S_N^M(y) = \sum_{k=0}^{[N/M]} \frac{(-N)_{Mk}}{k!} A_{N,k} y^k, \quad N = 0, 1, 2, \dots \quad (24)$$

where M is an arbitrary positive integer and the coefficients $A_{N,k} (N, k \geq 0)$ are arbitrary constants, real or complex and $(\lambda)_N$ is the Pochhammer symbol.

P -transform or pathway transform (pathway type fractional integral operators) is defined and represented in Dilip Kumar ([1], p. no. 56) and Kumar and Kilbas [2] as

Definition 1 : Let $\tau \in C, \delta, \beta \in R^+, \eta < 1$ and $x > 0$, then a type-1P-transform of a function $f(x)$ is defined as

$$(P_{\tau}^{\delta, \beta, \eta} f)(x) = \int_0^{\infty} Z_{\delta, \beta}^{\tau, \eta} (xt) f(t) dt \quad (25)$$

where $Z_{\delta, \beta}^{\tau, \eta} (x)$ denotes the function

$$Z_{\delta, \beta}^{\tau, \eta} (x) = \int_0^{\frac{1}{1-\eta}} \left(\frac{1}{1-\eta}\right)^{\frac{1}{\delta}} y^{\tau-1} [1 - (1-\eta)y^{\delta}]^{\frac{1}{1-\eta}} e^{-xy^{-\beta}} dy$$

Definition 2 : Let $\tau \in C$, $\beta \in R^+$, $\delta \in R$, $\delta \neq 0$, $\eta > 1$ and $x > 0$, then a type-2 P -transform of a function $f(x)$ is defined as

$$(P_{\tau}^{\delta, \beta, \eta+} f)(x) = \int_0^{\infty} Z_{\delta, \beta}^{\tau, \eta+}(xt) f(t) dt \tag{26}$$

where $Z_{\delta, \beta}^{\tau, \eta+}(x)$ denotes the function

$$Z_{\delta, \beta}^{\tau, \eta+}(x) = \int_0^{\infty} y^{\tau-1} [1 + (\eta - 1)y^{\delta}]^{-\frac{1}{\eta-1}} e^{-xy^{-\beta}} dy$$

The P -transform of both type are defined in the $L_{\tau, r}(0, \infty)$ space which contains the Lebesgue measurable complex valued functions f for which

$$\|f\|_{\tau, r} = \left(\int_0^{\infty} |t^{\tau} f(t)|^r \frac{dt}{t} \right)^{\frac{1}{r}} < \infty$$

for $1 \leq r < \infty$, $\tau \in C$. The P -transform of both types is obtained by using the pathway model of Mathai [4], Mathai and Haubold [5]. If $\beta = 1$, $\eta \rightarrow 1$, we have

$$\lim_{\eta \rightarrow 1-} Z_{\delta, 1}^{\tau, \eta-}(x) = \lim_{\eta \rightarrow 1+} Z_{\delta, 1}^{\tau, \eta+}(x) = Z_{\delta}^{\tau}(x) \tag{27}$$

where $Z_{\delta}^{\tau}(x)$ is the Kratzel function defined as

$$Z_{\delta}^{\tau}(x) = \int_0^{\infty} t^{\tau-1} e^{-t^{\delta}-xt^{-1}} dt, \quad x \geq 0$$

Now we are recalling the following lemmas which give the power function formulas for the above P -transform (pathway type fractional integral operators). These power function formulas (see [1]) are required for our present study.

Lemma 1 : Let $\tau, \rho \in C$, $\delta, \beta \in R^+$, $\eta < 1$ and $\Re(\rho) > 0$ be such that $-\Re(\tau) < \beta\Re(\rho)$, then the following formula holds (see [1], p. no. 82, eq. 3.52)

$$(P_{\tau}^{\delta, \beta, \eta-} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma\left(\frac{\tau + \beta\rho}{\delta}\right)\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta(1-\eta)^{\frac{\tau + \beta\rho}{\delta}}\Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta\rho}{\delta}\right)} \frac{1}{x^{\rho}} \tag{28}$$

Lemma 2 : Let $\tau, \rho \in C, \beta \in R^+, \delta \in R, \delta \neq 0, \eta > 1$ and $\Re(\rho) > 0$ be such that $-\Re(\tau) < \beta\Re(\rho) < \Re(\frac{\delta}{\eta-1}) - \Re(\tau)$ when $\delta > 0$ and $\Re(\frac{\delta}{\eta-1}) - \Re(\tau) < \beta\Re(\rho) < -\Re(\tau)$ when $\delta < 0$, then the following formula holds (see [1], p. no. 84, eq. 3.53)

$$(P_{\tau}^{\delta, \beta, \eta+} t^{\rho-1})(x) = \frac{\Gamma(\rho)\Gamma\left(\frac{\tau+\beta\rho}{\delta}\right)\Gamma\left(\frac{1}{\eta-1} - \frac{\tau+\beta\rho}{\delta}\right)}{|\delta|(\eta-1)^{\frac{\tau+\beta\rho}{\delta}}\Gamma\left(\frac{1}{\eta-1}\right)} \frac{1}{x^{\rho}} \quad (29)$$

2. Main Results :

2.1 Pathway type fractional integration of the product of Srivastava’s polynomial, multivariable Mittag-Leffler function and modified I-function of two variables.

In this section, we shall establish two theorems involving the composition of Srivastava’s polynomial, multivariable Mittag-Leffler function and modified I-function with the pathway type fractional integral operators.

Theorem 1 : Let $\tau, \rho \in C, \delta, \beta, a, b, c, f, \lambda, m_j \in R^+$ for $j = 1, 2, \dots, l$ and $\eta < 1$, then

$$\left\{ P_{\tau}^{\delta, \beta, \eta-} \left(t^{\rho-1} S_N^M (\sigma t^{\lambda}) E_{\mu_i, c}^{\lambda_i} (\sigma_1 t^{m_1}, \dots, \sigma_l t^{m_l}) I [at^c, bt^f] \right) \right\} (x) \\ = \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta x^{\rho} (1-\eta)^{\tau+\beta\rho/\delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta\lambda/\delta}} \right)^k \\ \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i/\delta}} \right)^{k_i} \times I_{p+2, q+1; V}^{m, n+2; U} \left[\begin{matrix} ax^{-c}(1-\eta)^{-\beta c/\delta} \\ bx^{-f}(1-\eta)^{-\beta f/\delta} \end{matrix} \middle| \begin{matrix} X_1, X_2, A_1; A_2; A_3; A_4 \\ B_1, Y_1; B_2; B_3; B_4 \end{matrix} \right] \quad (30)$$

where

$$X_1 = \left(1 - \rho - \lambda k - \sum_{i=1}^l m_i k_i; c, f; 1 \right) \quad (31)$$

$$X_2 = \left(\frac{\delta - \tau - \beta\rho - \beta\lambda k - \sum_{i=1}^l \beta m_i k_i}{\delta}; \frac{\beta c}{\delta}, \frac{\beta f}{\delta}; 1 \right) \quad (32)$$

$$Y_1 = \left(-\frac{1}{1-\eta} - \frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i}{\delta}; \frac{\beta c}{\delta}, \frac{\beta f}{\delta}; 1 \right) \quad (33)$$

Proof : In order to prove the equation (30), we first express the general class of polynomial and multivariable Mittag-Leffler function in series form with the help of equations (25) and (23) and expressing the modified *I*-function of two variables in terms of Mellin-Barnes contour integral with the help of the equation (1) and changing the order of integration, which is permissible under the given conditions, we get

$$\begin{aligned} \Delta = & \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \sigma^k \frac{\sigma_1^{k_1}}{k_1!} \dots \frac{\sigma_l^{k_l}}{k_l!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \theta_1(s_1) \theta_2(s_2) \\ & \times \alpha^{s_1} b^{s_2} \left(P_{\tau}^{\delta, \beta, \eta} - t^{\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + ds_2 - 1} \right) (x) ds_1 ds_2 \end{aligned} \quad (34)$$

Now using the power function formula (28), we get

$$\begin{aligned} P_{\tau}^{\delta, \beta, \eta} - t^{\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} &= x^{-\left(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right)} \\ & \times \frac{\Gamma\left(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right) \Gamma\left(\frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right) \Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta(1-\eta)^{\frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}} \Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right)} \end{aligned} \quad (35)$$

Now putting the equation (35) in the equation (34), we get

$$\begin{aligned} \Delta = & \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta x^{\rho} (1-\eta)^{\tau + \beta\rho/\delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta\lambda/\delta}} \right)^k \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i/\delta}} \right)^{k_i} \\ & \times \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \frac{\Gamma\left(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right) \Gamma\left(\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2/\delta\right)}{\Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right)} \end{aligned}$$

$$\times \theta_1(s_1)\theta_2(s_2) \left(\frac{ax^{-c}}{(1-\eta)^{\beta c/\delta}}\right)^{s_1} \left(\frac{bx^{-f}}{(1-\eta)^{\beta f/\delta}}\right)^{s_2} ds_1 ds_2 \tag{36}$$

Now interpreting the above equation (36) with Mellin-Barnes contour integral in terms of modified *I*-function of two variables with the help of the equation (1), we get the right of the equation (30).

Theorem 2 : Let $\tau, \rho \in C, \beta, a, b, c, f, \lambda, m_j \in R^+, \delta \in R$ for $j = 1, 2, \dots, l$ such that $\delta \neq 0$ and $\eta > 1$, then

$$\left\{ P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho-1} S_N^M(\sigma t^{\lambda}) E_{\mu_i, c}^{\lambda_i} \left(\sigma_1 t^{m_1}, \dots, \sigma_l t^{m_l} \right) I \left[at^c, bt^f \right] \right) \right\} (x)$$

$$\frac{1}{|\delta| x^{\rho} (\eta-1)^{\tau+\beta\rho/\delta} \Gamma(1/\eta-1)} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(\eta-1)^{\beta\lambda/\delta}} \right)^k$$

$$\prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(\eta-1)^{\beta m_i/\delta}} \right)^{k_i} \times I_{p+2, q+1; V}^{m+1, n+2; U} \left[\begin{matrix} ax^{-c} (1-\eta)^{-\beta c/\delta} \\ bx^{-f} (1-\eta)^{-\beta f/\delta} \end{matrix} \middle| \begin{matrix} X_1, X_2, A_1; A_2; A_3; A_4 \\ Y_2, b_j; B_1; B_2; B_3; B_4 \end{matrix} \right] \tag{37}$$

where X_1, X_2 are already given in the equations (31), (32) respectively and

$$Y_2 = \left(\frac{1}{\eta-1} - \frac{\left(\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i \right)}{\delta}, \frac{\beta c}{\delta}, \frac{\beta f}{\delta}; 1 \right) \tag{38}$$

3. Integral transforms :

In this section, we shall obtain some integral transforms like Beta transform, Laplace transform, Verma transform, Hankel transform and *K*-transform of the fractional integral formulas obtained in the previous section.

3.1 Beta transform :

Definition : The beta transform of a function $f(z)$ is defined as (see [12])

$$B\{f(z); s, p\} = \int_0^1 z^{s-1} (1-z)^{p-1} f(z) dz \tag{39}$$

Theorem 3 : Let $\tau, \rho \in C, \delta, \beta, a, b, c, f, \lambda, m_j \in R^+$ for $j = 1, 2, \dots, l$ and $\eta < 1$, then

$$\begin{aligned}
 & B \left\{ P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho-1} S_N^M (\sigma(zt)^\lambda) E_{\mu_i, c}^{\lambda_i} (\sigma_1(zt)^{m_1}, \dots, \sigma_l(zt)^{m_l}) I \left[\begin{matrix} a(zt)^c \\ b(zt)^f \end{matrix} \right] \right) (x) : s, p \right\} \\
 &= \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right) \Gamma(p)}{\delta x^\rho (1-\eta)^{\tau+\beta\rho/\delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta\lambda/\delta}} \right)^k \\
 & \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i/\delta}} \right)^{k_i} \times I_{p+3, q+2; V}^{m, n+3; U} \left[\begin{matrix} ax^{-c} (1-\eta)^{-\beta c/\delta} \\ bx^{-f} (1-\eta)^{-\beta f/\delta} \end{matrix} \middle| \begin{matrix} X_1, X_2, X_3, A_1; A_2; A_3; A_4 \\ B_1, Y_1, Y_2; B_2; B_3; B_4 \end{matrix} \right] \quad (40)
 \end{aligned}$$

where X_1, X_2, Y_1 are already given in the equations (31), (32) and (33) respectively and

$$X_3 = \left(1 - s - \lambda k - \sum_{i=1}^l m_i k_i; c, f; 1 \right) \quad (41)$$

$$Y_3 = \left(1 - s - p - \lambda k - \sum_{i=1}^l m_i k_i; c, f; 1 \right) \quad (42)$$

Proof : In order to prove the equation (40), using the definition (39) of beta transform, we get

$$\begin{aligned}
 & B \left\{ P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho-1} S_N^M (\sigma(zt)^\lambda) E_{\mu_i, c}^{\lambda_i} (\sigma_1(zt)^{m_1}, \dots, \sigma_l(zt)^{m_l}) I \left[a(zt)^c, b(zt)^f \right] \right) (x) : s, p \right\} \\
 &= \int_0^1 z^{s-1} (1-z)^{p-1} \left\{ P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho-1} S_N^M (\sigma(zt)^\lambda) E_{\mu_i, c}^{\lambda_i} (\sigma_1(zt)^{m_1}, \dots, \sigma_l(zt)^{m_l}) \right. \right. \\
 & \quad \left. \left. \times I \left[a(zt)^c, b(zt)^f \right] \right) (x) \right\} dz \quad (43)
 \end{aligned}$$

Now expressing the general class of polynomial and multivariable Mittag-Leffler function in series form with the help of equations (25) and (23) and expressing the modified I -function of two variables in terms of Mellin-Barnes contour integral with the help of equation (1) and changing the order of integration, which is permissible under the given conditions, we get

$$\begin{aligned}
&= \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu, i, c}^{\lambda_i} \sigma^k \frac{\sigma_1^{k_1}}{k_1!} \dots \frac{\sigma_l^{k_l}}{k_l!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \theta_1(s_1) \theta_2(s_2) \\
&\quad \times a^{s_1} b^{s_2} \left[\int_0^1 z^{s+\lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} (1-z)^{p-1} dz \right] \\
&\quad \left(P_{\tau}^{\delta, \beta, \eta, -} \cdot t^{\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} \right) (x) ds_1 ds_2 \tag{44}
\end{aligned}$$

Now using the power function formula (28), we get

$$\begin{aligned}
&= \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta x^{\rho} (1-\eta)^{\tau + \beta \rho / \delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu, i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta \lambda / \delta}} \right)^k \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i / \delta}} \right)^{k_i} \\
&\times \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \frac{\Gamma\left(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right) \Gamma\left(\tau + \beta \rho + \beta \lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2 / \delta\right)}{\Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta \rho + \beta \lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right)} \\
&\times \left[\int_0^1 z^{s+\lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} (1-z)^{p-1} dz \right] \theta_1(s_1) \theta_2(s_2) \left(\frac{ax^{-c}}{(1-\eta)^{\beta c / \delta}} \right)^{s_1} \left(\frac{bx^{-f}}{(1-\eta)^{\beta f / \delta}} \right)^{s_2} ds_1 ds_2 \tag{45}
\end{aligned}$$

Now evaluating the z -integral and substituting in the equation (45), we get

$$\begin{aligned}
&= \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right) \Gamma(p)}{\delta x^{\rho} (1-\eta)^{\tau + \beta \rho / \delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu, i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta \lambda / \delta}} \right)^k \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i / \delta}} \right)^{k_i} \\
&\times \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \frac{\Gamma\left(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right) \Gamma\left(\tau + \beta \rho + \beta \lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2 / \delta\right)}{\Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta \rho + \beta \lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right)}
\end{aligned}$$

$$\times \frac{\Gamma\left(s+\lambda k+\sum_{i=1}^l m_i k_i+c s_1+f s_2\right)}{\Gamma\left(s+p+\lambda k+\sum_{i=1}^l m_i k_i+c s_1+f s_2\right)} \theta_1\left(s_1\right) \theta_2\left(s_2\right)\left(\frac{a x^{-c}}{(1-\eta)^{\beta c / \delta}}\right)^{s_1}\left(\frac{b x^{-f}}{(1-\eta)^{\beta f / \delta}}\right)^{s_2} d s_1 d s_2 \quad (46)$$

Now interpreting the above equation (46) with the help of the equation (1) in terms of Mellin-Barnes contour integral, we get the required result (40).

Theorem 4 : Let $\tau, \rho \in C, \beta, a, b, c, f, \lambda, m_j \in R^+, \delta \in R$ for $j = 1, 2, \dots, l$ such that $\delta \neq 0$ and $\eta > 1$, then

$$B\left\{P_{\tau}^{\delta, \beta, \eta+}\left(t^{\rho-1} S_N^M\left(\sigma(z t)^{\lambda}\right) E_{\mu, c}^{\lambda_i}\left(\sigma_1(z t)^{m_1}, \dots, \sigma_l(z t)^{m_l}\right) I_{p, q; p_1, q_1; p_2, q_2; p_3, q_3}^{m, n; m_1, n_1; m_2, n_2; m_3, n_3}\left[\begin{array}{c} a(z t)^c \\ b(z t)^f \end{array}\right](x): s, p\right\}=\frac{\Gamma(p)}{|\delta| x^p(\eta-1)^{\tau+\beta \rho / \delta} \Gamma(1 / \eta-1)}$$

$$\sum_{k=0}^{[N / M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{M k}}{k!} A_{N, k} F_{\mu, c}^{\lambda_i}\left(\frac{\sigma x^{-\lambda}}{(\eta-1)^{\beta \lambda / \delta}}\right)^k \prod_{i=1}^l \frac{1}{k_i!}\left(\frac{\sigma_i x^{-m_i}}{(\eta-1)^{\beta m_i / \delta}}\right)^{k_i}$$

$$\times I_{p+3, q+2; V}^{m+1, n+3; U}\left[\begin{array}{c} a x^{-c}(1-\eta)^{-\beta c / \delta} \\ b x^{-f}(1-\eta)^{-\beta f / \delta} \end{array} \left| \begin{array}{c} X_1, X_2, X_3, A_1; A_2; A_3; A_4 \\ Y_2, B_1, Y_3; B_2; B_3; B_4 \end{array} \right.\right] \quad (47)$$

where X_1, X_2, X_3, Y_2, Y_3 are already given in the equations (31), (32), (41), (38) and (42) respectively.

3.2 Laplace transform :

Definition : The Laplace transform of a function $f(z)$ is defined as (see [6], [11])

$$L\{f(z)\}=\int_0^{\infty} e^{-q z} f(z) d z \quad (48)$$

Theorem 5 : Let $\tau, \rho \in C, \delta, \beta, a, b, c, f, \lambda, m_j \in R^+$ for $j = 1, 2, \dots, l$ and $\eta < 1$, then

$$L\left\{P_{\tau}^{\delta, \beta, \eta-}\left(t^{\rho-1} S_N^M\left(\sigma(z t)^{\lambda}\right) E_{\mu, c}^{\lambda_i}\left(\sigma_1(z t)^{m_1}, \dots, \sigma_l(z t)^{m_l}\right) I\left[\begin{array}{c} a(z t)^c \\ b(z t)^f \end{array}\right](x)\right\}$$

$$\begin{aligned}
 &= \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta qx^\rho (1-\eta)^{\tau+\beta\rho/\delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma(qx)^{-\lambda}}{(1-\eta)^{\beta\lambda/\delta}} \right)^k \\
 &\prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i(qx)^{-m_i}}{(1-\eta)^{\beta m_i/\delta}} \right)^{k_i} \times I_{p+3, q+1; V}^{m, n+3; U} \left[\begin{matrix} a(qx)^{-c} (1-\eta)^{-\beta c/\delta} \\ b(qx)^{-f} (1-\eta)^{-\beta f/\delta} \end{matrix} \middle| \begin{matrix} X_1, X_2, X_4, A_1; A_2; A_3; A_4 \\ B_1, Y_1; B_2; B_3; B_4 \end{matrix} \right] \quad (49)
 \end{aligned}$$

where X_1, X_2, Y_1 are already given in the equations (31), (32) and (33) respectively and

$$X_4 = \left(-\lambda k - \sum_{i=1}^l m_i k_i; c, f; 1 \right) \quad (50)$$

Proof : In order to prove the equation (49), using the definition (48) of Laplace transform, we get

$$\begin{aligned}
 &L \left\{ P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho-1} S_N^M(\sigma(z)t^\lambda) E_{\mu_i, c}^{\lambda_i}(\sigma_1(z)^{m_1}, \dots, \sigma_l(z)^{m_l}) I[a(z)^c, b(z)^f] \right) (x) \right\} \\
 &= \int_0^{\infty} e^{-qz} \left\{ P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho-1} S_N^M(\sigma(z)t^\lambda) E_{\mu_i, c}^{\lambda_i}(\sigma_1(z)^{m_1}, \dots, \sigma_l(z)^{m_l}) \right. \right. \\
 &\quad \left. \left. \times I[a(z)^c, b(z)^f] \right) (x) \right\} dz \quad (51)
 \end{aligned}$$

Now expressing the general class of polynomial and multivariable Mittag-Leffler function in series form with the help of equations (25) and (23) and expressing the modified I -function of two variables in terms of Mellin-Barnes contour integral with the help of the equation (1) and changing the order of integration, which is permissible under the given conditions, we get

$$\begin{aligned}
 &= \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \sigma^k \frac{\sigma_1^{k_1}}{k_1!} \dots \frac{\sigma_l^{k_l}}{k_l!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \theta_1(s_1) \theta_2(s_2) \times a^{s_1} b^{s_2} \\
 &\left[\int_0^{\infty} e^{-qz} z^{\lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2} dz \right] \left(P_{\tau}^{\delta, \beta, \eta} t^{\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} \right) (x) ds_1 ds_2 \quad (52)
 \end{aligned}$$

Now using the power function formula (28), we get

$$\begin{aligned}
 &= \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta x^\rho (1-\eta)^{\tau+\beta\rho/\delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta\lambda/\delta}} \right)^k \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i/\delta}} \right)^{k_i} \\
 &\times \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \frac{\Gamma\left(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right) \Gamma\left(\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2/\delta\right)}{\Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right)} \\
 &\times \left[\int_0^\infty e^{-qz} z^{1+\lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} dz \right] \theta_1(s_1) \theta_2(s_2) \left(\frac{ax^{-c}}{(1-\eta)^{\beta c/\delta}} \right)^{s_1} \left(\frac{bx^{-f}}{(1-\eta)^{\beta f/\delta}} \right)^{s_2} ds_1 ds_2 \quad (53)
 \end{aligned}$$

Now evaluating the z-integral and substituting in the above equation (53), we get

$$\begin{aligned}
 &= \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta x^\rho (1-\eta)^{\tau+\beta\rho/\delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta\lambda/\delta}} \right)^k \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i/\delta}} \right)^{k_i} \\
 &\times \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \frac{\Gamma\left(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right) \Gamma\left(\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2/\delta\right)}{\Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right)} \\
 &\times \frac{\Gamma\left(1 + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right)}{q^{\lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 + 1}} \theta_1(s_1) \theta_2(s_2) \left(\frac{ax^{-c}}{(1-\eta)^{\beta c/\delta}} \right)^{s_1} \left(\frac{bx^{-f}}{(1-\eta)^{\beta f/\delta}} \right)^{s_2} ds_1 ds_2 \quad (54)
 \end{aligned}$$

Now interpreting the above equation (54) with the help of the equation (1) in terms of Mellin-Barnes contour integral, we get the required result (49).

Theorem 6 : Let $\tau, \rho \in C, \beta, a, b, c, f, \lambda, m_j \in R^+, \delta \in R$ for $j = 1, 2, \dots, l$ such that $\delta \neq 0$, and $\eta > 1$, then

$$\begin{aligned}
 & L \left\{ P_{\tau}^{\delta, \beta, \eta+} \left(t^{\rho-1} S_N^M (\sigma(z)t)^{\lambda} E_{\mu_i, c}^{\lambda_i} \left(\sigma_1(z)t^{m_1}, \dots, \sigma_l(z)t^{m_l} \right) I \left[\begin{matrix} a(z)t^c \\ b(z)t^f \end{matrix} \right] \right) (x) \right\} \\
 &= \frac{1}{|\delta| q x^{\rho} (\eta-1)^{\tau+\beta\rho/\delta} \Gamma(1/\eta-1)} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma(qx)^{-\lambda}}{(\eta-1)^{\beta\lambda/\delta}} \right)^k \\
 & \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i(qx)^{-m_i}}{(\eta-1)^{\beta m_i/\delta}} \right)^{k_i} \times I_{p+3, q+1; V}^{m+1, n+3; U} \left[\begin{matrix} a(qx)^{-c} (\eta-1)^{-\beta c/\delta} \\ b(qx)^{-f} (\eta-1)^{-\beta f/\delta} \end{matrix} \middle| \begin{matrix} X_1, X_2, X_4, A_1; A_2: A_3; A_4 \\ Y_2, B_1; B_2: B_3; B_4 \end{matrix} \right] \quad (55)
 \end{aligned}$$

where X_1, X_2, X_4, Y_2 are already given in the equations (31), (32), (50) and (38) respectively.

3.3 Verma transform :

Definition : The Verma transform of a function $f(z)$ is defined as (see [6], [12])

$$V\{f(z)\} = \int_0^{\infty} (sz)^{q-1} e^{-1/2sz} W_{\kappa, \nu}(sz) f(z) dz \quad (56)$$

where $W_{\kappa, \nu}(z)$ represents Whittaker function.

Theorem 7 : Let $\tau, \rho \in C, \delta, \beta, a, b, c, f, \lambda, m_j \in R^+$ for $j = 1, 2, \dots, l$ and $\eta < 1$, then

$$\begin{aligned}
 & V \left\{ P_{\tau}^{\delta, \beta, \eta-} \left(t^{\rho-1} S_N^M (\sigma(z)t)^{\lambda} E_{\mu_i, c}^{\lambda_i} \left(\sigma_1(z)t^{m_1}, \dots, \sigma_l(z)t^{m_l} \right) I \left[\begin{matrix} a(z)t^c \\ b(z)t^f \end{matrix} \right] \right) (x) \right\} \\
 &= \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right) s^{-q}}{\delta x^{\rho} (1-\eta)^{\tau+\beta\rho/\delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma(sx)^{-\lambda}}{(1-\eta)^{\beta\lambda/\delta}} \right)^k \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i(sx)^{-m_i}}{(1-\eta)^{\beta m_i/\delta}} \right)^{k_i} \\
 & \times I_{p+4, q+2; V}^{m, n+4; U} \left[\begin{matrix} a(sx)^{-c} (1-\eta)^{-\beta c/\delta} \\ b(sx)^{-f} (1-\eta)^{-\beta f/\delta} \end{matrix} \middle| \begin{matrix} X_1, X_2, X_5, X_6, A_1; A_2: A_3; A_4 \\ B_1, Y_1, Y_4; B_2: B_3; B_4 \end{matrix} \right] \quad (57)
 \end{aligned}$$

where X_1, X_2, Y_1 are already given in the equations (31), (32) and (33) respectively and

$$X_5 = \left(-\frac{1}{2} - q - \nu - \lambda k - \sum_{i=1}^l m_i k_i; c, f; 1 \right) \quad (58)$$

$$X_6 = \left(-\frac{1}{2} - q + v - \lambda k - \sum_{i=1}^l m_i k_i; c, f; 1 \right) \tag{59}$$

$$Y_4 = \left(\kappa - q - \lambda k - \sum_{i=1}^l m_i k_i; c, f; 1 \right) \tag{60}$$

Proof : In order to prove the equation (57), using the definition (56) of Verma transform, we get

$$\begin{aligned} & V \left\{ P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho-1} S_N^M (\sigma(zt)^\lambda) E_{\mu_i, c}^{\lambda_i} \left(\sigma_1(zt)^{m_1}, \dots, \sigma_l(zt)^{m_l} \right) I \left[a(zt)^c, b(zt)^f \right] \right) (x) \right\} \\ &= \int_0^\infty z^{q-1} e^{-\frac{1}{2}sz} W_{\kappa, \nu}(sz) \left\{ P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho-1} S_N^M (\sigma(zt)^\lambda) E_{\mu_i, c}^{\lambda_i} \left(\sigma_1(zt)^{m_1}, \dots, \sigma_l(zt)^{m_l} \right) \right. \right. \\ &\quad \left. \left. \times I \left[a(zt)^c, b(zt)^f \right] \right) (x) \right\} dz \tag{61} \end{aligned}$$

Now expressing the general class of polynomial and multivariable Mittag-Leffler function in series form with the help of equations (25) and (23) and expressing the modified I -function of two variables in terms of Mellin-Barnes contour integral with the help of the equation (1) and changing the order of integration, which is permissible under the given conditions, we get

$$\begin{aligned} &= \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N, k} F_{\mu_i, c}^{\lambda_i} \sigma^k \frac{\sigma_1^{k_1}}{k_1!} \dots \frac{\sigma_l^{k_l}}{k_l!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \theta_1(s_1) \theta_2(s_2) \\ &\quad \times a^{s_1} b^{s_2} \left[\int_0^\infty z^{q+\lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} e^{-\frac{1}{2}sz} W_{\kappa, \nu}(sz) dz \right] \\ &\quad \left(P_{\tau}^{\delta, \beta, \eta} \left(t^{\rho+\lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} \right) (x) \right) ds_1 ds_2 \tag{62} \end{aligned}$$

Now using the power function formula (28), we get

$$= \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta x^\rho (1-\eta)^{\tau+\beta\rho/\delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N, k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta\lambda/\delta}} \right)^k \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i/\delta}} \right)^{k_i}$$

$$\begin{aligned}
& \times \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2) \Gamma(\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2 / \delta)}{\Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right)} \\
& \times \left[\int_0^1 z^{q + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2 - 1} e^{-\frac{1}{2}sz} W_{\kappa, \nu}(sz) dz \right] \theta_1(s_1) \theta_2(s_2) \\
& \left(\frac{ax^{-c}}{(1-\eta)^{\beta c / \delta}} \right)^{s_1} \left(\frac{bx^{-f}}{(1-\eta)^{\beta f / \delta}} \right)^{s_2} ds_1 ds_2 \tag{63}
\end{aligned}$$

Now evaluating the z -integral and substituting in the above equation (63), we get

$$\begin{aligned}
& = \frac{\Gamma\left(\frac{1}{1-\eta} + 1\right)}{\delta x^\rho (1-\eta)^{\tau + \beta\rho / \delta}} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma x^{-\lambda}}{(1-\eta)^{\beta\lambda / \delta}} \right)^k \prod_{i=1}^l \frac{1}{k_i!} \left(\frac{\sigma_i x^{-m_i}}{(1-\eta)^{\beta m_i / \delta}} \right)^{k_i} \\
& \times \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \Phi(s_1, s_2) \frac{\Gamma(\rho + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2) \Gamma(\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2 / \delta)}{\Gamma\left(\frac{1}{1-\eta} + 1 + \frac{\tau + \beta\rho + \beta\lambda k + \sum_{i=1}^l \beta m_i k_i + \beta cs_1 + \beta fs_2}{\delta}\right)} \\
& \times \frac{\Gamma\left(\frac{1}{2} + \nu + q + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right) \Gamma\left(\frac{1}{2} - \nu + q + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right)}{\Gamma\left(1 - \kappa + q + \lambda k + \sum_{i=1}^l m_i k_i + cs_1 + fs_2\right)} \\
& \times \theta_1(s_1) \theta_2(s_2) \left(\frac{ax^{-c}}{(1-\eta)^{\beta c / \delta}} \right)^{s_1} \left(\frac{bx^{-f}}{(1-\eta)^{\beta f / \delta}} \right)^{s_2} ds_1 ds_2 \tag{64}
\end{aligned}$$

Now interpreting the above equation (64) with the help of the equation (1) in terms of Mellin-Barnes contour integral, we get the required result (57).

Theorem 8 : Let $\tau, \rho \in C, \beta, a, b, c, f, \lambda, m_j \in R^+, \delta \in R$ for $j = 1, 2, \dots, l$ such that $\delta \neq 0$, and $\eta > 1$, then

$$\begin{aligned}
 & V \left\{ P_{\tau}^{\delta, \beta, \eta+} \left(t^{\rho-1} S_N^M (\sigma(zt)^\lambda) E_{\mu_i, c}^{\lambda_i} \left(\sigma_1(zt)^{m_1}, \dots, \sigma_l(zt)^{m_l} \right) I \left[a(zt)^c, b(zt)^f \right] \right) (x) \right\} \\
 &= \frac{s^{-q}}{|\delta|x^\rho(\eta-1)^{\tau+\beta\rho/\delta} \Gamma(1/\eta-1)} \sum_{k=0}^{[N/M]} \sum_{k_1, \dots, k_l=0}^{\infty} \frac{(-N)_{Mk}}{k!} A_{N,k} F_{\mu_i, c}^{\lambda_i} \left(\frac{\sigma(sx)^{-\lambda}}{(\eta-1)^{\beta\lambda/\delta}} \right)^k \prod_{i=1}^l \\
 & \frac{1}{k_i!} \left(\frac{\sigma_i(sx)^{-m_i}}{(\eta-1)^{\beta m_i/\delta}} \right)^{k_i} \times I_{p+4, q+2; V}^{m+1; U} \left[\begin{matrix} a(sx)^{-c} (\eta-1)^{-\beta c/\delta} \\ b(sx)^{-f} (\eta-1)^{-\beta f/\delta} \end{matrix} \middle| \begin{matrix} X_1, X_2, X_5, X_6, A_1; A_2: A_3; A_4 \\ Y_2, B_1, Y_4; B_2: B_3; B_4 \end{matrix} \right] \quad (65)
 \end{aligned}$$

where $X_1, X_2, X_5, X_6, Y_2, Y_4$ are already given in the equations (31), (32), (58), (59), (38) and (60) respectively.

4. Conclusion :

Our results are very general in nature. In this paper, we have given two theorems involving the composition of Srivastava's polynomial, multivariable Mittag-Leffler function and modified I -function of two variables with pathway type fractional integral operators and ten theorems for Beta, Laplace, and Verma transforms of the composition of Srivastava's polynomial, multivariable Mittag-Leffler function and modified I -function of two variables with pathway type fractional integral operators. On specializing parameters, the general class of polynomial, multivariable Mittag-Leffler function and modified I -function of two variables can be reduced to several more simpler special functions. Therefore all the results of this paper can also be obtain for more simpler special functions.

References :

- [1] Kumar, Dilip. (2014) : Ph.D thesis on Applications of pathway model and special functions in applied analysis and reaction rate theory in astrophysics, Anna University, Chennai.
- [2] Kumar, Dilip and Kilbas, A.A. (2010) : Fractional Calculus of P -transform, Fac. Calc. and Appl. Anal., 13, No. 3, 309-327.
- [3] Kumari, S., Vasudevan Nambisan and Rathi, A.K. (2012) : A Study of I -function of two variables, airXiv:1212.6717v1[math.CV].

- [4] Mathai, A.M. (2005) : A pathway to matrix-variate gamma and normal densities, Linear algebra Appl., 396, 317-328.
- [5] Mathai, A.M. and Haubold, H.J. (2007) : Pathway model, super statistics, Tsallis statistics and a generalized measure of entropy, Physica A 375, 110-122.
- [6] Mathai, A.M., Saxena, R.K. and Haubold, H.J. (2010) : The H -function : theory and applications, Springer, New York.
- [7] Mittag-Leffler, G.M. (1903) : Sur la nouvelle fonction $E_\alpha(x)$, C.R. Acad. Sci. Paris, 137, 554-558.
- [8] Prabhakar, T.R. (1971) : A singular integral equation with a generalized Mittag-Leffler function kernel, Yokohama Math. J., 19, 7-15.
- [9] Prasad, Y.N. and Prasad, S. (1979) : J. of Scientific research, Banaras Hindu University, 67-76.
- [10] Saxena, R.K., Kalla, S.K. and Saxena, R. (2011) : Multivariable analogue of Mittag-Leffler function, integral transform and special functions, Vol. 7, 533-548.
- [11] Schiff, I.L. (1999) : The Laplace transform, Theory and Applications, Springer, New York.
- [12] Sneddon, I.N. (1979) : The use of integral transforms, Tata McGraw-Hill, New Delhi.
- [13] Srivastava, H.M. (1972) : A contour integral involving Fox's H -function, Indian J. Math., 14, 1-6.
- [14] Wiman, A. (1905) : Uber de fundamental satz in der theorie de funktionen $E_\alpha(x)$, Acta Math, 29, 191-201.