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Finiteness, uniqueness and isomorphic properties in decompositions of group rings *R*[*G*]

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Abstract:

We have found that any finite decompositions of group rings R[G] algebraic structure depend upon three properties. These properties are finiteness, uniqueness as well as isomorphism. Without utilizing these three properties we cannot decompose any algebraic structure finitely. We have first used many lemmas, propositions and theorems to find out the decompositions in group rings R[G]. Let us suppose that G be a group. As we decompose it as a finite number of two decompositions, as follow.

$$G \cong H_1 \times H_2 \times H_3 \times H_4 \times \ldots\ldots. \times H_l$$

$$G \cong K_1 \times K_2 \times K_3 \times K_4 \times \ldots\ldots.. \times K_t$$

Here we can write as, $H_1 \cong K_1$, $H_2 \cong K_2$ *,....,* $H_l \cong K_t$ *and* $l = t$.

Group G is isomorphic to its product of sub groups. Also, each subgroup of one decomposition becomes isomorphic to the corresponding decomposition of other subgroup. This also clears the uniqueness of decompositions of the same group G. Similarly, we can decompose a group rings structure R[G]. These decompositions of group rings R[G] can be done as follow,

$$R[G] \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times........\times M_{n_t}(D_t)$$

$$R[G] \cong N_{n_1}(D_1) \times N_{n_2}(D_2) \times........\times N_{n_k}(D_k)$$

Here each $M_{n_t}(D_t) \cong N_{n_k}(D_k)$ *for each t and k and also* $t = k$.

This shows that decomposition of group rings R[G] is unique, finite and isomorphic. Throughout this paper we have discussed upon uniqueness, finiteness as well as isomorphic property of group rings R[G].

Keywords: Decomposition of $R[G]$, finiteness, uniqueness, isomorphic, semi-simple, division rings, ascending chain condition (A.C.C.), descending chain condition (D.C.C.)

1. Introduction:

A group G is said to be indecomposable if $G \neq \{e\}$ and G is not direct product of two of its proper sub groups.

1.1 Theorem: If G satisfies either ascending chain condition (A.C.C.) or descending chain condition (D.C.C.), then group G is a direct product of a finite numbers of indecomposable proper subgroups.

Proof: If group G is indecomposable then it is already done. If $G = H \times K$ and H and K are indecomposable subgroups of G then it is also done. But if group G is not direct product of a finite numbers of indecomposable subgroups, then G is not indecomposable. Therefore $G = H_1 \times K_1$ but at least one of H_1 and K_1 should be decomposable. Without the loss of generality, we suppose that K_1 is decomposable, so we decompose K_1 as $K_1 = H_2 \times K_2$ but here at least one of H_2 and K_2 is decomposable. We repeat the process and get,

$$G = H_1 \times (H_2 \times (H_3 \times ....) ....) \cong H_1 \times H_2 \times H_3 \times ....$$

Hence, there is an infinite strictly ascending chain and infinite strictly descending chain of group G. These are as

$$H_1 < H_1 \times H_2 < H_1 \times H_2 \times H_3 < \ldots \ldots, \text{ and}$$

$$G > K_1 > K_2 > K_3 > \ldots\ldots.$$

These violate both A.C.C. and D.C.C. Therefore, group G is a direct product of a finite numbers of indecomposable subgroups.

1.2 Proposition: M is an indecomposable A-module if and only if $\text{End}_A(M)$ is local as A is a semi-simple ring.

1.3 Proposition: Let M is an artinian and noetherian $R[G]$ - module. Then M is indecomposable if and only if $\text{End}_{R[G]}(M)$ is a local ring.

1.4 Lemma: If $f: M \rightarrow N$, $g: N \rightarrow M$ satisfy, $gf = 1_M$ then N splits as $\text{im}(f) \oplus \ker(g)$ and f is injective therefore M is a direct summand of N.

1.5 Theorem: (Krull-Schmidt Theorem) Let M be an artinian and noetherian module, if we decompose

$M \cong U_1 \oplus U_2 \oplus U_3 \oplus .... \oplus U_k$ as well as

$M \cong V_1 \oplus V_2 \oplus V_3 \oplus .... \oplus V_l$, then $k = l$ and there is a permutation σ of $\{1, 2, 3, \ldots., k\}$ such than

$U_i \cong V_{\sigma(i)}$ for all $i = 1, 2, 3, ..., k$.

1.6 Proposition: Let $R[G]$ be a group ring and M is a $R[G]$-module. Then the following are equivalent

(i) M is semi-simple.

(ii) M is the sum of a family of simple sub-module.

(iii) M is direct sum of a family of simple sub-modules.

1.7 Lemma: Every sub-module and every quotient module of a semi-simple module is semi-simple.

1.8 Lemma: Any non-zero semi-simple module contains a simple module.

2. Theorem: Let $R[G]$ is a ring and is semi-simple artinian, if and only if there exist a finite number of division rings D_1, D_2, …., D_t and positive integers, n_1, n_2,……, n_t such that,

$$R[G] \cong \prod_{i=1}^{t} M_{n_i}(D_i)$$

Proof: We will prove that a ring of the form $R[G] \cong \prod_{i=1}^{t} M_{n_i}(D_i)$ is semi-simple artinian. So we will show that $M_n(D)$ is a simple ring. Let D be a division ring. It is simple; hence every ideal of $M_n(D)$ is of the form $M_n(I)$ for an ideal I of D. Let V be the n-tuple row space D^n. The ring $M_n(D)$ acts on the right by matrix multiplication so we suppose V as a right $M_n(D)$-module.

By elementary linear algebra, V is a simple right $M_n(D)$-module. So, we establish the direct sum decomposition.

$M_n(D) = R_1 \oplus R_2 \oplus R_3 \oplus \ldots\ldots \oplus R_n$, here R_i is the right ideal of $M_n(D)$ consisting of matrices, all of whose rows are zero except for the i-th one. As a right $M_n(D)$-module, every R_i is isomorphic to V and hence $M_n(D) \cong nV$. This shows that the ring $M_n(D)$ is right semi-simple.

Now let $R_1, R_2, \ldots.., R_n$ be right semi-simple artinian rings and let $R[G]$ be their direct product. Now, we can write, $R_i = I_{i_1} \oplus I_{i_2} \oplus I_{i_3} \oplus \ldots\ldots \oplus I_{i_m}$ as a sum of simple right ideals. We suppose that R_i as an ideal in ring $R[G]$, then every I_{ij} will be taken as a simple right ideal of ring $R[G]$.

Now from, $(R[G])_R = R_1 \oplus R_2 \oplus \ldots\ldots \oplus R_n = \bigoplus_{i,j} I_{i,j}$

Thus, we conclude that $R[G]$ is right semi-simple.

Now let $R[G]$ be a right semi-simple artinian ring, we decompose $R[G]_R$ into a finite sum of simple right ideals. So, we will write,

$$R[G]_R = n_1 V_1 \oplus n_2 V_2 \oplus \ldots\ldots.. \oplus n_t V_t \qquad (1)$$

Here $V_1, V_2, \ldots\ldots, V_t$ are mutually non-isomorphic simple $R[G]$-module. Now we will find the endomorphism ring of the two $R[G]$ - modules in equation (1).

Thus, we have, $\mathrm{End}_R(R[G]) = \mathrm{End}_R(n_1 V_1 \oplus n_2 V_2 \oplus \ldots\ldots.. \oplus n_t V_t)$

Now we get a ring isomorphism as,

$$R[G] \cong \mathrm{End}_R(n_1 V_1) \times \ldots\ldots.. \times \mathrm{End}_R(n_t V_t)$$

$$R[G] \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \ldots\ldots.. \times M_{n_t}(D_t)$$

3. Theorem: Let A is semi-simple ring. It is algebra over an algebraically closed field K. Let M be an A - module finite dimensional over K and if

$M = U_1 \oplus U_2 \oplus \ldots\ldots.. \oplus U_n$, $U_1, U_2, \ldots\ldots, U_n$ all indecomposable.

$M = V_1 \oplus V_2 \oplus \ldots\ldots.. \oplus V_m$, $V_1, V_2, \ldots\ldots, U_m$ are indecomposable then $m = n$ and $U_i \cong V_i$ up to permutation.

Proof: As there are two decompositions

$$M \cong U_1 \oplus U_2 \oplus \ldots\ldots.. \oplus U_n$$
$$M \cong V_1 \oplus V_2 \oplus \ldots\ldots.. \oplus V_m$$

Let us suppose that, $\pi_i \in \mathrm{End}_A(M)$ be the projection onto U_i and ρ_j be the projection onto V_j. Then we will consider, $\pi_i \rho_j | U_1$. This is either invertible or nilpotent, but

$$\sum_j \pi_1 \rho_j | U_1 = \sum \pi_1 1_M | U_1 = {}^1 U_1$$

As $\pi_1 \rho_j | U_1 \in \mathrm{End}_A(U_1)$ for each j, and $\mathrm{End}_A(U_1)$ is local. But not all of them are nilpotent. Now, without the loss of generality, we may assume that $\pi_1 \rho_j | U_1$ is invertible with inverse, $\theta \in \mathrm{End}_A(U_1)$.

We have the composition,

$$\rho_1 | U_1 \pi_1 | V_1 \theta$$

$$U_1 \to V_1 \to U_1 \to U_1$$

Then $\beta \mathrm{o} \alpha = {}^1U_1$ by the definition of θ. As we claim that,

$$V_1 \cong \mathrm{im}(\alpha) \oplus \ker(\beta), \text{ so if } x \in \mathrm{im}(\alpha) \cap \ker(\beta) \text{ and } x = \alpha y.$$

Hence, $y = \beta\alpha y = \beta x = 0$.

Again, if $Z \in V_1$ then we may write

$$\begin{aligned} &Z = (Z - \alpha\beta Z) + \alpha\beta Z \\ &\in \ker\beta \in \mathrm{im}\alpha \end{aligned}$$

Since V_1 is indecomposable and $\mathrm{im}(\alpha) \neq 0$, so we conclude that $\ker(\beta) = 0$, so α, β are isomorphism, $U_1 \cong V_1$. Now we can say that,

$$U_1(V_2 \oplus V_3 \oplus \ldots\ldots. \oplus V_m) = 0$$

That is because if,

$x \in U_1 \cap (V_2 + V_3 + \ldots\ldots. + V_m)$, then $x = \beta\alpha x$ and α is a restriction of ρ_1, which annihilates $V_2, V_3, \ldots., V_m$.

Therefore,

$$M = U_1 + V_2 + V_3 + \ldots.. + V_m \text{ and the sum is direct and } U_1 \cong V_1, \text{ so}$$

$$M = U_1 \oplus V_2 \oplus V_3 \oplus \ldots.. \oplus V_m$$

We have done by induction, considering the decomposition that,

$$U_2 \oplus U_3 \oplus \ldots.. \oplus U_n \cong M/U_1 \cong V_2 \oplus V_3 \oplus \ldots.. \oplus V_m$$

4. Conclusions:

Decomposition of finite group rings into indecomposable modules is very important because it possesses three properties. (i). Finiteness (ii). Uniqueness (iii). Isomorphism. We have also found that, if the decompositions of group rings $R[G]$ are done into indecomposable matrix division rings then these decompositions follow, Finiteness, Uniqueness and Isomorphic properties as, follow into matrix division rings.

$$R[G] \cong M_{n_1}(D_1) \times M_{n_2}(D_2) \times \ldots\ldots.. \times M_{n_t}(D_t) \text{ and}$$

$$R[G] \cong N_{n_1}(D_1) \times N_{n_2}(D_2) \times \ldots\ldots.. \times N_{n_k}(D_k)$$

are two decompositions.

Then we have each $M_{n_t}(D_t)$ is isomorphic to each $N_{n_k}(D_k)$ of other corresponding decomposition, also $t=k$. Thus, we have seen that these three properties/conditions can be used to examine the finiteness of decompositions of any algebraic structures in abstract algebra. Therefore, decomposition of any algebraic structure proves that one can decompose an algebraic structure upto the smallest sub-algebraic structures. After that we can't decompose the sub-algebraic structures. It is also provided that any two decompositions of an algebraic structure possess the exchange property as well as isomorphic property upto suitable permutations.

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