The Mathematics Education **ISSN 0047-6269**
Volume - LV, No. 1, March 2021
Refereed and Peer-Reviewed Quarterly Journal
Journal website : www.internationaljournalsiwan.com

Incomplete Multivariable Aleph-function and Integral of Three Parameters Calculus

by **Frédéric Ayant,**
Teacher in High School, France
E-mail : fredericayant@gmail.com
&
Prvindra Kumar,
Department of Mathematics,
D.J. College, Baraut - 250611, India
E-mail : prvindradjc@gmail.com

Abstract :

In the present paper, we defined the incomplete multivariable aleph-function. We will calculate an integral depending of several parameters concerning this new function. At the end, we shall see several corollaries and remarks.

Keywords : Incomplete Gamma function, incomplete multivariable aleph-function, incomplete multivariable I-function, incomplete Aleph-function of two variables, incomplete I-function of two variables multiple Mellin-Barnes integrals contour, incomplete double H-function, incomplete H-function.

2010 Mathematics Subject Classification : 33C05, 33C60.

1. Introduction and Preliminaries :

Srivastava et al. [17] have studied the incomplete Gamma-function and incomplete hypergeometric function. More recently, Srivastava et al. [21] have introduced and studied the incomplete H-function and the incomplete $\overline{H}$-function. More recently, several researchers, Bansal et al. [3,5,6], Bansal and Kumar [4], Bansal and Choi [2] have introduced and studied the incomplete Aleph-function, the incomplete I-function and calculate the integrals about the incomplete H-function and gave applications respectively. The aim of this document is to define the incomplete of the multivariable Aleph-functions defined by Ayant [1]. The multivariable Aleph-function is an extension of the multivariable H-function defined by Srivastava and Panda [19, 20] and the Aleph-function studied by Sudland [22]. The incomplete multivariable Aleph-function is a generalization of the multivariable Aleph-function cited above and the incomplete Aleph-function defined by Bansal et al. [5]. We will see an application to calculate an integral and we will give several particular cases.

The incomplete Gamma functions $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ are defined in the following manner :

$$\gamma(\alpha, x) = \int_0^x u^{\alpha-1} e^{-u}\, du \quad (\Re(\alpha) > 0; x \geq 0). \tag{1.1}$$

$$\Gamma(\alpha, x) = \int_x^\infty u^{\alpha-1} e^{-u}\, du \quad (x \geq 0; \Re(\alpha) > 0 \text{ when } x = 0). \tag{1.2}$$

These incomplete $\gamma(\alpha, x)$ and $\Gamma(\alpha, x)$ satisfy the following decomposition formula :

$$\gamma(\alpha, x) + \Gamma(\alpha, x) = \Gamma(\alpha)\, (\mathrm{Re}(\alpha) > 0). \tag{1.3}$$

In this paper, x is a positive real number.

First, we define and we note the incomplete Gamma Aleph-function as follows :

We have : $${}^{(\Gamma)}\aleph(z_1,\ldots,z_r) = {}^{(\Gamma)}\aleph^{0,n:m_1,n_1,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\right|$$

$$\left[\left(a_1;\alpha_1^{(1)},\ldots,\alpha_1^{(r)},x\right)\right],\left[\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)\right]_{n+1,p_i}:$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad ,\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)\right]_{m+1,q_i}:$$

$$\left.\begin{matrix}\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)\right]_{1,n_1},\left[\tau_{i(1)}\left(c_{ji(1)}^{(1)},\gamma_{ji(1)}^{(1)}\right)\right]_{n_1+1,p_i^{(1)}};..;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)\right]_{1,n_r},\left[\tau_{i(r)}\left(c_{ji(r)}^{(r)},\gamma_{ji(r)}^{(r)}\right)\right]_{n_r+1,p_i^{(r)}} \\ \left[\left(d_j^{(1)},\delta_j^{(1)}\right)\right]_{1,m_1+1},\left[\tau_{i(1)}\left(d_{ji(1)}^{(1)},\delta_{ji(1)}^{(1)}\right)\right]_{m_1+1,q_i^{(1)}};..;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)\right]_{1,m_r},\left[\tau_{i(r)}\left(d_{ji(r)}^{(r)},\delta_{ji(r)}^{(r)}\right)\right]_{m_r+1,q_i^{(r)}}\end{matrix}\right)$$

$$= \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,...,s_r)\prod_{k=1}^{r}\zeta_k(s_k)\,z_k^{s_k}\,ds_1.....ds_r \tag{1.4}$$

with $\omega = \sqrt{-1}$

$$\psi(s_1,...,s_r) = \frac{\Gamma\left(1-a_1+\sum_{k=1}^{r}\alpha_1^{(k)}s_k,x\right)\prod_{j=2}^{n}\Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k\right)}{\sum_{i=1}^{R}\left[\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k\right)\right]} \tag{1.5}$$

and $$\zeta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)\prod_{j=1}^{n_k}\Gamma\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k\right)\right]} \tag{1.6}$$

where $j = 1$ to r and $k = 1$ or r. a's, b's, c's and d's are complex numbers, and the α's, β's, γ's and δ's are assumed to be positive real numbers for standardization purpose such that

$$U_i^{(k)} = \sum_{j=1}^{n}\alpha_j^{(k)} + \tau_i\sum_{j=n+1}^{p_i}\alpha_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} + \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_i^{(k)}}\gamma_{ji^{(k)}}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} - \sum_{j=1}^{m_k}\delta_j^{(k)}$$

$$-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_i^{(k)}}\delta_{ji^{(k)}}^{(k)} \le 0 \tag{1.7}$$

The reals numbers τ_i are positives for $i = 1$ to R, $\tau_{i^{(k)}}$ are positives for $i^{(k)} = 1$ to $R^{(k)}$

The contour L_k is in the s_k-p lane and run from $\sigma - i\infty$ to $\sigma + i\infty$ where σ is a real number with loop, if necessary, ensure that the poles of $\Gamma(d_j^{(k)} - \delta_j^{(k)}s_k)$ with

$j = 1$ to m_k are separated from those of $\Gamma(1- a_j + \sum_{i=1}^{r} \alpha_j^{(k)} s_k)$ with $j = 1$ to n and $\Gamma(1- c_j^{(k)} + \gamma_j^{(k)} s_k)$ with $j = 1$ to n_k to the left of the contour L_k. The condition for absolute convergence of multiple Mellin-Barnes type contour (1.9) can be obtained by extension of the corresponding conditions for multivariable H-function given by as :

$$|\arg z_k| < \frac{1}{2} A_i^{(k)} \pi, \text{ where}$$

$$A_i^{(k)} = \sum_{j=1}^{n} \alpha_j^{(k)} - \tau_i \sum_{j=n+1}^{p_i} \alpha_{ji}^{(k)} - \tau_i \sum_{j=1}^{q_i} \beta_{ji}^{(k)} + \sum_{j=1}^{n_k} \gamma_j^{(k)} - \tau_i^{(k)} \sum_{j=n_k+1}^{p_i(k)} \gamma_{ji(k)}^{(k)} + \sum_{j=1}^{m_k} \delta_j^{(k)}$$

$$- \tau_{i(k)} \sum_{j=m_k+1}^{q_i(k)} \delta_{ji(k)}^{(k)} > 0, \text{ with } k = 1, \ldots, r,\ i = 1, \ldots, R,\ i^{(k)} = 1, \ldots, R^{(k)} \qquad (1.8)$$

The complex numbers z_i are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function.

We may establish the asymptotic behavior (see B.L.J. Braaksma [7] in the following convenient form :

$$^{(\Gamma)}\aleph(z_1, \ldots, z_r) = 0\,(|z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r}), \max(|z_1|, \ldots, |z_r|) \to 0$$

$$^{(\Gamma)}\aleph(z_1, \ldots, z_r) = 0\,(|z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r}), \min(|z_1|, \ldots, |z_r|) \to \infty$$

where, with $k = 1, \ldots, r$: $\alpha_k = \min[\mathrm{Re}(d_j^{(k)} / \delta_j^{(k)})], j = 1, \ldots, m_k$ and

$$\beta_k = \max[\mathrm{Re}((c_j^{(k)} - 1) / \gamma_j^{(k)})],\ j = 1, \ldots, n_k$$

We will use these following notations in this paper

$$U = p_i, q_i, \tau_i ; R ; V = m_1, n_1 ; \ldots ; m_r, n_r : U_{1,1} = p_i + 1, q_i + 1, \tau_i ; R \qquad (1.9)$$

$$W = p_{i(1)}, q_{i(1)}, \tau_{i(1)} ; R^{(1)}, \ldots, p_{i(r)}, q_{i(r)}, \tau_{i(r)} ; R^{(r)} \qquad (1.10)$$

$$A = \left\{\left(a_j ; \alpha_j^{(1)}, \ldots, \alpha_j^{(r)}\right)\right\}_{2,n}, \left\{\tau_i\left(a_{ji} ; \alpha_{ji}^{(1)}, \ldots, \alpha_{ji}^{(r)}\right)\right\}_{n+1,p_i} \qquad (1.11)$$

$$B=\left\{\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)\right\}_{m+1,q_i} \tag{1.12}$$

$$C=\left\{\tau_{i(1)}\left(c_j^{(1)};\gamma_j^{(1)}\right)\right\}_{1,n_1},\left\{\tau_{i(1)}\left(c_{ji(1)}^{(1)};\gamma_{ji(1)}^{(1)}\right)\right\}_{n_1+1,p_{i(1)}},\ldots,\left\{\left(c_j^{(r)};\gamma_j^{(r)}\right)\right\}_{1,n_r},$$
$$\left\{\tau_{i(r)}\left(c_{ji(r)}^{(r)};\gamma_{ji(r)}^{(r)}\right)\right\}_{n_r+1,p_{i(r)}} \tag{1.13}$$

$$D=\left\{\left(d_j^{(1)};\delta_j^{(1)}\right)\right\}_{1,m_1},\left\{\tau_{i(1)}\left(d_{ji(1)}^{(1)};\delta_{ji(1)}^{(1)}\right)\right\}_{m_1+1,q_{i(1)}},\ldots,\left\{\left(d_j^{(r)};\delta_j^{(r)}\right)\right\}_{1,m_r},$$
$$\left\{\tau_{i(r)}\left(d_{ji(r)}^{(r)};\delta_{ji(r)}^{(r)}\right)\right\}_{m_r+1,q_{i(r)}} \tag{1.14}$$

Now, we defined the analogue incomplete gamma multivariable Aleph-function :

We have : $^{(\gamma)}\aleph(z_1,\ldots,z_r)={}^{(\gamma)}\aleph^{0,n:m_1,n_1,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i(1)},q_{i(1)},\tau_{i(1)};R^{(1)};\ldots;p_{i(r)},q_{i(r)};\tau_{i(r)};R^{(r)}}\left(\begin{matrix} z_1\\ \cdot\\ \cdot\\ z_r\end{matrix}\right|$

$$\left[\left(a_1;\alpha_1^{(1)},\ldots,\alpha_1^{(r)},x\right)\right],\left[\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)\right]_{2,n},\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)\right]_{n+1,p_i}:$$

$$\ldots\ldots\ldots\ldots\ldots\ldots,\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)\right]_{m+1,q_i}:$$

$$\left[\left(c_j^{(1)},\gamma_j^{(1)}\right)\right]_{1,n_1},\left[\tau_{i(1)}\left(c_{ji(1)}^{(1)},\gamma_{ji(1)}^{(1)}\right)\right]_{n_1+1,p_i^{(1)}};\ldots;\left[\left(c_j^{(r)},\gamma_j^{(r)}\right)\right]_{1,n_r},\left[\tau_{i(r)}\left(c_{ji(r)}^{(r)},\gamma_{ji(r)}^{(r)}\right)\right]_{n_r+1,p_i^{(r)}}$$
$$\left.\left[\left(d_j^{(1)},\delta_j^{(1)}\right)\right]_{1,m_1+1},\left[\tau_{i(1)}\left(d_{ji(1)}^{(1)},\delta_{ji(1)}^{(1)}\right)\right]_{m_1+1,q_i^{(1)}};\ldots;\left[\left(d_j^{(r)},\delta_j^{(r)}\right)\right]_{1,m_r},\left[\tau_{i(r)}\left(d_{ji(r)}^{(r)},\delta_{ji(r)}^{(r)}\right)\right]_{m_r+1,q_i^{(r)}}\right)$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi'(s_1,\ldots,s_r)\prod_{k=1}^{r}\zeta_k(s_k)\,z_k^{s_k}\,ds_1\ldots\ldots ds_r \tag{1.15}$$

with $\omega=\sqrt{-1}$

$$\psi'(s_1,\ldots,s_r)=\frac{\gamma\left(1-a_1+\sum_{k=1}^{r}\alpha_1^{(k)}s_k,x\right)\prod_{j=2}^{n}\Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k\right)}{\sum_{i=1}^{R}\left[\tau_i\prod_{j=n+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k\right)\right]} \tag{1.16}$$

$$\text{and } \zeta_k(s_k) = \frac{\prod_{j=1}^{m_k} \Gamma\left(d_j^{(k)} - \delta_j^{(k)} s_k\right) \prod_{j=1}^{n_k} \Gamma\left(1 - c_j^{(k)} + \gamma_j^{(k)} s_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}} \left[\tau_{i^{(k)}} \prod_{j=m_k+1}^{q_{i^{(k)}}} \Gamma\left(1 - d_{ji^{(k)}}^{(k)} + \delta_{ji^{(k)}}^{(k)} s_k\right) \prod_{j=n_k+1}^{p_{i^{(k)}}} \Gamma\left(c_{ji^{(k)}}^{(k)} - \gamma_{ji^{(k)}}^{(k)} s_k\right)\right]} \quad (1.17)$$

where $j = 1$ to r and $k = 1$ or r.

We have the same mathematical data and formulas that the Gamma incomplete multivariable Aleph-function.

By using the utilisation of the relation (1.3), I's easy to show the decomposition formula concerning the incomplete multivariable Aleph-functions.

$$^{(\Gamma)}\aleph(z_1,\ldots,z_r) + {}^{(\gamma)}\aleph(z_1,\ldots,z_r) = \aleph(z_1,\ldots,z_r) \quad (1.18)$$

In the following section, we give the integral formula. This integral will be used later.

2. Required integral :

We have the formula, (Y.A. Brychkov [8], Ch. 4.1.3, Eq. 12, p. 119). This integral involves the hyperbolic sinus function.

$$\int_0^a x^s (a-x)^{s+\frac{1}{2}} \sinh(b\sqrt{4x(a-x)})\, dx = 2^{-2s-\frac{3}{2}} \sqrt{\pi} a^{2s+2}\, b \frac{\Gamma(2s+\frac{5}{2})}{\Gamma(2s+3)}$$

$$ {}_1F_2 \left(\begin{matrix} 2s+\frac{5}{2} \\ \cdot \\ \frac{3}{2}, 2s+3 \end{matrix} \middle| \frac{ab^2}{8} \right) \quad (2.1)$$

where $\mathrm{Re}(s) > -\frac{5}{4}$

In the following section, we give two general relations.

3. Main results :

Using the integral defined above at the incomplete Gamma multivariable Aleph-function. We note $X = x(a-x)$.

Theorem 1 : $\int_0^a x^s(a-x)^{s+\frac{1}{2}}\,{}^{(\Gamma)}\aleph(z_1X^{\epsilon_1},\dots,z_rX^{\epsilon_r})\sinh(b\sqrt{4x(a-x)})\,dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$

$$\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\Gamma)}\aleph^{0,n+1:V}_{U_{11}:W}\left(\begin{array}{c|c} z_1a^{\epsilon_1} & (a_1;\alpha_1,\dots,\alpha_1^{(r)},x),A_1,A:C \\ \cdot & \cdot \\ \cdot & \cdot \\ z_ra^{\epsilon_r} & B,B_1:D \end{array}\right) \tag{3.1}$$

Where

$$A_1=\left(-\tfrac{3}{2}-2s-n\,;2\epsilon_1,\dots,2\epsilon_r\right);\ B_1=\left(-2-2s-n\,;2\epsilon_1,\dots,2\epsilon_r\right) \tag{3.2}$$

provided $\epsilon_i>0$; $i=1,\dots,r$, $\mathrm{Re}\,(s)+\sum_{i=1}^{r}\min_{1\le j\le m_i}\epsilon_i\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>-\frac{5}{4}$. The conditions given by the equations (1.7) and (1.8) are verified.

Proof : We note L the left hand side of the equation (3.1). First time, we replace the Gamma incomplete multivariable Aleph-function by this multiple integrals contour defined by (1.4), this gives :

$$L=\int_0^a x^s(a-x)^{s+\frac{1}{2}}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\dots,s_r)\prod_{k=1}^{r}\zeta_i(s_i)\,z_i^{s_i}\,x^{\epsilon_is_i}(a-x)^{\epsilon_is_i}$$

$$\sinh(b\sqrt{4x(a-x)})\,ds_1\dots ds_r\,dx \tag{3.3}$$

Interchanging the order of the integrals, which is justifiable due to absolute convergence of the integral involved in the process, this gives :

$$L=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\dots,s_r)\prod_{i=1}^{r}\zeta_i(s_i)\,z_i^{s_i}\int_0^a x^{s+\sum_{i=1}^r\epsilon_is_i}(a-x)^{s+\sum_{i=1}^r\epsilon_is_i+\frac{1}{2}}$$

$$\sinh(b\sqrt{4x(a-x)})\,dx\,ds_1\dots ds_r \tag{3.4}$$

Using the lemma, we get :

$$L=2^{-\frac{3}{2}}\sqrt{\pi}a^2b\,\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\dots,s_r)\prod_{i=1}^{r}\zeta_i(s_i)z_i^{s_i}\,a^{\sum_{i=1}^r\epsilon_is_i}\,\frac{\Gamma\left(2s+2\sum_{i=1}^r\epsilon_is_i+\frac{5}{2}\right)}{\Gamma\left(2s+2\sum_{i=1}^r\epsilon_is_i+3\right)}$$

$$
{}_1F_2\left(\begin{array}{c} 2s+2\sum_{i=1}^{r}\in_i s_i+\frac{5}{2} \\ \cdot \\ \frac{3}{2},\, 2s+2\sum_{i=1}^{r}\in_i s_i+3 \end{array}\middle|\ \frac{ab^2}{8}\right) ds_1\dots ds_r \tag{3.5}
$$

Now, we use the definition of the hypergeometric function (see Slater [16]) and interchanging the n-series and the multiple $(s_1,\dots,s_r)$-integrals (because we have the absolute convergence of the integral involved in the process), this gives :

$$
L = 2^{-\frac{3}{2}}\sqrt{\pi}a^2 b \sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\dots,s_r)\prod_{i=1}^{r}\zeta_i(s_i)z_i^{s_i}
$$

$$
a^{\sum_{i=1}^{r}\in_i s_i}\frac{\Gamma\left(2s+2\sum_{i=1}^{r}\in_i s_i+\frac{5}{2}\right)\left(2s+2\sum_{i=1}^{r}\in_i s_i+\frac{5}{2}\right)_n}{\Gamma\left(2s+2\sum_{i=1}^{r}\in_i s_i+3\right)\left(2s+2\sum_{i=1}^{r}\in_i s_i+3\right)_n} ds_1\dots ds_r \tag{3.6}
$$

Applying the following property $\Gamma(a)(a+n)=\Gamma(a+n)$, $a\neq 0,-1,-2,\dots$, we obtain

$$
L = 2^{-\frac{3}{2}}\sqrt{\pi}a^2 b \sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,\dots,s_r)\prod_{i=1}^{r}\zeta_i(s_i)z_i^{s_i}
$$

$$
a^{\sum_{i=1}^{r}\in_i s_i}\frac{\Gamma\left(2s+2\sum_{i=1}^{r}\in_i s_i+\frac{5}{2}+n\right)}{\Gamma\left(2s+2\sum_{i=1}^{r}\in_i s_i+3+n\right)} ds_1\dots ds_r \tag{3.7}
$$

Interpreting the above multiple integrals contour by the Gamma incomplete multivariable Aleph-function, we obtain the desired formula.

We have the similar result concerning the incomplete Gamma multivariable Aleph-function.

Theorem 2 : $\int_0^a x^s(a-x)^{s+\frac{1}{2}}\,{}^{(\gamma)}\aleph(z_1X^{\in_1},\dots,z_rX^{\in_r})\,\sinh(b\sqrt{4x(a-x)})\,dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2 b$

$$
\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\gamma)}\aleph^{0,n+1:V}_{U_{11}:W}\left(\begin{array}{c} z_1a^{\in_1} \\ \cdot \\ \cdot \\ z_ra^{\in_r} \end{array}\middle|\begin{array}{c} (a_1;\alpha_1,\dots,\alpha_1^{(r)},x),A_1,A:C \\ \cdot \\ \cdot \\ B,B_1:D \end{array}\right) \tag{3.8}
$$

under the same conditions and notation that the theorem 1. The proof is similar.

In the four section, we gives several corollaries and remarks.

4. Special Cases :

First taking $\tau_i, \tau_{i(1)},...,\tau_{i(r)} \rightarrow 1$, the incomplete multivariable Aleph-functions reduce to Gamma incomplete of the multivariable I-function defined Sharma and Ahmad [14], we have the two results.

Corollary 1 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}}\,{}^{(\Gamma)}I(z_1X^{\in_1},...,z_rX^{\in_r})\ \sinh(b\sqrt{4x(a-x)})\ dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\Gamma)}I^{0,n+1:V}_{U_1:W_1}\left(\begin{array}{c} z_1a^{\in_1} \\ \cdot \\ \cdot \\ z_ra^{\in_r}\end{array}\middle| \begin{array}{c}(a_1;\alpha_1,...,\alpha_1^{(r)},x),A_1,A^1:C^1 \\ \cdot \\ \cdot \\ B^1,B_1:D^1\end{array}\right) \tag{4.1}$$

where

$$U_1 = p_i, q_i; R; V = m_1, n_1;...; m_r, n_r \tag{4.2}$$

$$W_1 = p_{i(1)}, q_{i(1)}; R^{(1)},..., p_{i(r)}, q_{i(r)}; R^{(r)} \tag{4.3}$$

$$A^1 = \left\{\left(a_j; \alpha_j^{(1)},..., \alpha_j^{(r)}\right)\right\}_{2,n}, \left\{\left(a_{ji}; \alpha_{ji}^{(1)},..., \alpha_{ji}^{(r)}\right)\right\}_{n+1,p_i} \tag{4.4}$$

$$B^1 = \left\{\left(b_{ji}; \beta_{ji}^{(1)},..., \beta_{ji}^{(r)}\right)\right\}_{m+1,q_i} \tag{4.5}$$

$$C^1 = \left\{\left(c_j^{(1)}, \gamma_j^{(1)}\right)\right\}_{1,n_1}, \left\{\left(c_{ji(1)}^{(1)}; \gamma_j^{(1)}\right)\right\}_{n_1+1,p_i(1)},......,$$
$$\left\{\left(c_j^{(r)}; \gamma_j^{(r)}\right)\right\}_{1,n_r}, \left\{\left(c_{ji(r)}^{(r)}; \gamma_{ji(r)}^{(r)}\right)\right\}_{n_r+1,p_i(r)} \tag{4.6}$$

$$D^1 = \left\{\left(d_j^{(1)}; \delta_j^{(1)}\right)\right\}_{1,m_1}, \left\{\left(d_{ji(1)}^{(1)}; \delta_{ji(1)}^{(1)}\right)\right\}_{m_1+1,q_i(1)},......,$$
$$\left\{\left(d_j^{(r)}; \delta_j^{(r)}\right)\right\}_{1,m_r}, \left\{\left(d_{ji(r)}^{(r)}; \delta_{ji(r)}^{(r)}\right)\right\}_{m_r+1,q_i(r)} \tag{4.7}$$

A_1 and B_1 are cited by the equations (3.2). We have the following result about the Gamma incomplete I-function.

Corollary 2 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}}\,{}^{(\gamma)}I(z_1X^{\in_1},\ldots,z_rX^{\in_r})\,\sinh(b\sqrt{4x(a-x)})\,dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\gamma)}I^{0,n+1:V}_{U_1:W_1}\left(\begin{array}{c|c} z_1a^{\in_1} & (a_1;\alpha_1,\ldots,\alpha_1^{(r)},x),A_1,A^1:C^1 \\ \cdot & \cdot \\ \cdot & \cdot \\ z_ra^{\in_r} & B^1,B_1:D^1 \end{array}\right) \tag{4.8}$$

under the same conditions concerning the incomplete multivariable Aleph-function (see the section 3) with $\tau_i,\ \tau_{i(1)},\ldots,\ \tau_{i(r)}\to 1$.

We suppose $\tau_i,\ \tau_{i(1)},\ldots,\ \tau_{i(r)}\to 1$ and $R^{(1)}=\ldots=R^{(r)}=1$, the incomplete multivariable Aleph-functions reduce to incomplete of multivariable H-function defined by Srivastava and Panda [19, 20], this gives.

Corollary 3 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}}\,{}^{(\Gamma)}H(z_1X^{\in_1},\ldots,z_rX^{\in_r})\,\sinh(b\sqrt{4x(a-x)})\,dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\Gamma)}H^{0,n+1:X}_{p+1,\,q+1:V}\left(\begin{array}{c|c} z_1a^{\in_1} & (a_1;\alpha_1,\ldots,\alpha_1^{(r)},x),A_1,A:C \\ \cdot & \cdot \\ \cdot & \cdot \\ z_ra^{\in_r} & B,B_1:D \end{array}\right) \tag{4.9}$$

where

$$X=m_1,n_1;\ldots;m_r,n_r;\ V=p_1,q_1;\ldots;p_r,q_r \tag{4.10}$$

$$A=\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{2,p}:B=\left(b_j;\beta_j^{(1)},\ldots,\beta_j^{(r)}\right)_{1,q} \tag{4.11}$$

$$C=\left(c_j^{(1)},\gamma_j^{(1)}\right)_{1,p_1};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,p_r} \tag{4.12}$$

$$D=\left(d_j^{(1)},\delta_j^{(1)}\right)_{1,q_1};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,q_r} \tag{4.13}$$

A_1 and B_1 are noted by the equation (3.2).

Corollary 4 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}} {}^{(\gamma)}H(z_1X^{\epsilon_1},...,z_rX^{\epsilon_r})\sinh(b\sqrt{4x(a-x)})\,dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\gamma)}H^{0,n+1:X}_{p+1,\,q+1:V}\left(\begin{array}{c|c} z_1a^{\epsilon_1} & (a_1;\alpha_1,...,\alpha_1^{(r)},x),A,A^1:C \\ \cdot & \cdot \\ \cdot & \cdot \\ z_ra^{\epsilon_r} & B,B_1:D \end{array}\right) \quad (4.14)$$

Under the conditions verified by the incomplete multivariable Aleph-function and $\tau_i, \tau_{i(1)}, ..., \tau_{i(r)} \to 1$ and $R^{(1)} = ... = R^{(r)} = 1$.

Let $r = 2$, we give two formulas concerning the incomplete Aleph-function of two variables defined by Sharma [13] and Kumar [10], we have the two following formulas.

Corollary 5 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}} {}^{(\Gamma)}\aleph(z_1X^{\epsilon_1}, z_2X^{\epsilon_2})\sinh(b\sqrt{4x(a-x)})\,dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\Gamma)}\aleph^{0,n+1:V}_{U_{11}:W}\left(\begin{array}{c|c} z_1a^{\epsilon_1} & (a_1;\alpha_1,\alpha_1^{(2)},x),A_1,A_2:C_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ z_2a^{\epsilon_2} & B_2,B_1:D_2 \end{array}\right) \quad (4.15)$$

the quantities A_2, B_2, C_2 and D_2 replace the quantities A, B, C and D respectively with $r = 2$ and

$$A_1 = (-\tfrac{3}{2} - 2s - n\,;\, 2\epsilon_1, 2\epsilon_2)\,;\, B_1 = (-2 - 2s - n\,;\, 2\epsilon_1, 2\epsilon_2) \quad (4.16)$$

Corollary 6 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}} {}^{(\gamma)}\aleph(z_1X^{\epsilon_1}, z_2X^{\epsilon_2})\sinh(b\sqrt{4x(a-x)})\,dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\gamma)}\aleph^{0,n+1:V}_{U_{11}:W}\left(\begin{array}{c|c} z_1a^{\epsilon_1} & (a_1;\alpha_1,\alpha_1^{(2)},x),A_1,A_2:C_2 \\ \cdot & \cdot \\ \cdot & \cdot \\ z_2a^{\epsilon_2} & B_2,B_1:D_2 \end{array}\right) \quad (4.17)$$

under the conditions verified by the theorems with $r = 2$.

Let τ_i, τ_i', $\tau_i'' \to 1$, the incomplete aleph-function of two variables are replaced by the incomplete I-function of two variables defined by Sharma and Mishra [15], we obtain :

Corollary 7 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}} {}^{(\Gamma)}I(z_1X^{\epsilon_1}, z_2X^{\epsilon_2}) \sinh(b\sqrt{4x(a-x)})\, dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty} \frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\Gamma)}I_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1a^{\epsilon_1} & (a_1;\alpha_1,\alpha_1^{(2)},x), A_1, A_2' : C_2' \\ \vdots & \vdots \\ z_2a^{\epsilon_2} & B_2', B_1 : D_2' \end{array}\right) \tag{4.18}$$

the quantities A_2', B_2', C_2' and D_2' replace the quantities A^1, B^1, C^1 and D^1 respectively with $r = 2$.

Corollary 8 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}} {}^{(\gamma)}I(z_1X^{\epsilon_1}, z_2X^{\epsilon_2}) \sinh(b\sqrt{4x(a-x)})\, dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty} \frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\gamma)}I_{U_{11}:W}^{0,n+1:V}\left(\begin{array}{c|c} z_1a^{\epsilon_1} & (a_1;\alpha_1,\alpha_1^{(2)},x), A_1, A_2' : C_2' \\ \vdots & \vdots \\ z_2a^{\epsilon_2} & B_2', B_1 : D_2' \end{array}\right) \tag{4.19}$$

Taking τ_i, τ_i', $\tau_i'' \to 1$ and $R = R' = R'' = 1$, the incomplete I-function of two variables reduces to incomplete H-function of two variables defined by Gupta and Mittal [9], this gives :

Corollary 9 :

$$\int_0^a x^s(a-x)^{s+\frac{1}{2}} {}^{(\Gamma)}H(z_1X^{\epsilon_1}, z_2X^{\epsilon_2}) \sinh(b\sqrt{4x(a-x)})\, dx = 2^{-\frac{3}{2}}\sqrt{\pi}a^2b$$

$$\sum_{n=0}^{\infty} \frac{1}{\left(\frac{3}{2}\right)^n n!}\left(\frac{ab^2}{8}\right)^n {}^{(\Gamma)}H_{p+1,\,q+1:V}^{0,n+1:X}\left(\begin{array}{c|c} z_1a^{\epsilon_1} & (a_1;\alpha_1,\alpha_1^{(2)},x),\, A_1, A\,; C \\ \vdots & \vdots \\ z_2a^{\epsilon_2} & B, B_1 : D \end{array}\right) \tag{4.20}$$

where, $X = m_1, n_1; m_2, n_2; V = p_1, q_1; p_2, q_2 : A = \left(a_j; \alpha_j^{(1)}, \alpha_j^{(2)}\right)_{2,p}$:

$$B = \left(b_j; \beta_j^{(1)}, \beta_j^{(2)}\right)_{1,q} \qquad (4.21)$$

$$C = \left(c_j^{(1)}, \gamma_j^{(1)}\right)_{1,p_1}; \left(c_j^{(2)}, \gamma_j^{(2)}\right)_{1,p_2}; D = \left(d_j^{(1)}, \delta_j^{(1)}\right)_{1,q_1}; \left(d_j^{(2)}, \delta_j^{(2)}\right)_{1,q_2} \qquad (4.22)$$

Corollary 10 :

$$\int_0^a x^s (a-x)^{s+\frac{1}{2}} {}^{(\gamma)}H(z_1 X^{\epsilon_1}, z_2 X^{\epsilon_2}) \sinh(b\sqrt{4x(a-x)})\, dx = 2^{-\frac{3}{2}} \sqrt{\pi} a^2 b$$

$$\sum_{n=0}^{\infty} \frac{1}{\left(\frac{3}{2}\right)^n n!} \left(\frac{ab^2}{8}\right)^n {}^{(\gamma)}H^{0,n+1:X}_{p+1,\, q+1:V} \left(\begin{array}{c|c} z_1 a^{\epsilon_1} & (a_1; \alpha_1, \alpha_1^{(2)}, x), A_1, A; C \\ \vdots & \vdots \\ z_2 a^{\epsilon_2} & B, B_1 : D \end{array} \right) \qquad (4.23)$$

under the same conditions concerning the incomplete multivariable I-function (see the corollaries 1 and 2) where $r = 2$.

Remarks :

If $r = 1$, the incomplete multivariable Aleph-function reduce to Incomplete of the aleph-function of one variable defined by Sudland [22], see Bansal et al. [5] for more precisions.

The incomplete multivariable I-function is replaced by the incomplete of the I-function of one variable defined by Saxena [12], see Bansal et al. [4] about this study.

The incomplete multivariable H-function reduce to incomplete of H-function of one variable (see Srivastava et al. [21]), see Bansal et al. [3], Bansal and Choi [2].

Srivastava et al. [21] have introduced and studied the incomplete $\overline{H}$-function. We can studied the incomplete of the I-function defined by Rathie [11].

5. Conclusion :

The importance of our all the results lies in their manifold generality. First, by specializing the various parameters as well as variable in the incomplete multivariable H-functions $^{(\Gamma)}\aleph()$ and $^{(\gamma)}\aleph()$, we obtain a large number of results involving remarkably wide variety of useful incomplete special functions (or product of such special functions) which are expressible in term of H-function defined by Bansal et al. [6], and hypergeometric function of one variable. Secondly, by specializing the parameters of these functions, we can get a large number of integrals about the incomplete multivariable incomplete special-functions of one or more variables. Thirdly, by specializing the parameters of the integral involving here, we can to obtain a large number of new integrals. These new functions have huge applications in physics, science, mechanics and other disciplines, see Bansal et al. [2, 5], fractional calculus Bansal and Kumar [3], probability law, see Bansal and Choi [2] (Pathway law).

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