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On unified infinite integral involving product of multivariable Gimel-function and others special functions

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Abstract :

In this paper, we evaluate a infinite integral whose integrand is the factor $z^{\upsilon-1}(z-1)^{-\mu}(z+1)^{-\rho}(a+bz^q)^{-\sigma}$. Next, with its help we establish the second infinite unified integral whose integrand involves the product of $(\tau,\ \beta)$-generalized associated Legendre function of first kind, ${}^{\tau,\beta}P_k^{m,n}(z)$, general class of polynomial $S_N^M(x)$, $\overline{H}$-function and multivariable Gimel-function. At the end of this study, we shall several particular cases and remarks.

Keywords : Multivariable Gimel-function, multiple integral contours, general class of polynomials, generalized associated Legendre function of first kind, $\overline{H}$-function, multivariable Aleph-function, multivariable I-functions, multivariable H-function.

2010 Mathematics Subject Classification : 33C99, 33C60, 44A20

1. Introduction and preliminaries :

Throughout this paper, let $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{N}$ be set of complex numbers, real numbers and positive integers respectively. Also $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$.

Recently Jain and Kumawat [8] have determined an infinite integrals involving generalized associated Legendre function of the first kind and multivariable H-function defined by Srivastava and Panda [15,16]. The aim of this paper, is to study an infinite unified integral involving generalized associated Legendre function of the first species, general class of polynomials, $\overline{H}$-function and multivariable Gimel function. In the first time, we give a brief definition of the multivariable Gimel-function.

The multivariable Gimel function [2] is an unified special function, it's an extension of the multivariable Aleph-function defined by Ayant [1], the multivariable I-function defined by Prasad [9], the multivariable I-function defined by Prathima et al. [11] at a time, of course this function is a generalization of the multivariable H-function. To define this function, we use the multiple Mellin-Barnes integrals contour. Its contracted form is the following

$$\Im(z_1,..,z_r) = \Im^{U;0,n_r:V}_{X;p_{i_r},q_{i_r},\tau_{i_r}:R_r:Y}\left(\begin{array}{c|c} z_1 & \mathbf{A}\,;\boldsymbol{A}:A \\ \vdots & \vdots \\ z_r & \mathbf{B}\,;\boldsymbol{B}:B \end{array}\right) = \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\psi(s_1,...,s_r)$$

$$\prod_{k=1}^{r}\theta_k(s_k)\, z_k^{s_k} ds_1 ... ds_r \tag{1.1}$$

with $\omega = \sqrt{-1}$

The following quantities $\psi(s_1,..., s_r)$, $\theta_k(s_k)$ $(k = 1,..., r)$ are defined by Ayant [2]. About this paper, we assume that the multivariable Gimel-function converges absolutely. For convergence conditions and other details (parameters) concerning the multivariable Gimel-function, see Ayant [2].

Following the lines of Braaksma ([3], p. 278), we may establish the asymptotic behavior in the following convenient form :

$$\Im(z_1,...,z_r)=0(|z_1|^{\alpha_1},...,|z_r|^{\alpha_r}),\ \max(|z_1|,...,|z_r|)\to 0$$

$$\Im(z_1,...,z_r)=0(|z_1|^{\beta_1},...,|z_r|^{\beta_r}),\ \min(|z_1|,...,|z_r|)\to\infty \text{ where } i=1,...,r:$$

$$\alpha_i=\min_{1\le j\le m^{(i)}}\operatorname{Re}\left[D_j^{(i)}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right] \text{ and } \beta_i=\max_{1\le j\le n^{(i)}}\operatorname{Re}\left[C_j^{(i)}\left(\frac{c_j^{(i)}-1}{\gamma_j^{(i)}}\right)\right]$$

In your investigation, we shall use the following notations.

$$\mathbf{A}=[(a_{2j};\alpha_{2j}^{(1)},\alpha_{2j}^{(2)};A_{2j})]_{1,n_2},[\tau_{i_2}(a_{2ji_2};\alpha_{2ji_2}^{(1)},\alpha_{2ji_2}^{(2)};A_{2ji_2})]_{n_2+1,p_{i_2}},$$
$$[(a_{3j};\alpha_{3j}^{(1)},\alpha_{3j}^{(2)},\alpha_{3j}^{(3)};A_{3j})]_{1,n_3},$$
$$[\tau_{i_3}(a_{3ji_3};\alpha_{3ji_3}^{(1)},\alpha_{3ji_3}^{(2)},\alpha_{3ji_3}^{(3)};A_{3ji_3})]_{n_3+1,p_{i_3}},....,[(a_{(r-1)j};\alpha_{(r-1)j}^{(1)},....,$$
$$\alpha_{(r-1)j}^{(r-1)};A_{(r-1)j})]_{1,n_{r-1}},[\tau_{i_{r-1}}(a_{(r-1)ji_{r-1}};\alpha_{(r-1)ji_{r-1}}^{(1)},....,$$
$$\alpha_{(r-1)ji_{r-1}}^{(r-1)};A_{(r-1)ji_{r-1}})]_{n_{r-1}+1,p_{i_{r-1}}} \tag{1.2}$$

$$\mathbf{A}=[(a_{rj};\alpha_{rj}^{(1)},...,\alpha_{rj}^{(r)};A_{rj})]_{1,n_r},[\tau_{i_r}(a_{rji_r};\alpha_{rji_r}^{(1)},...,\alpha_{rji_r}^{(r)};A_{rji_r})]_{n+1,p_{i_r}} \tag{1.3}$$

$$A=[(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})]_{1,n^{(1)}},[\tau_{i^{(1)}}(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)};C_{ji^{(1)}}^{(1)})]_{n^{(1)}+1,p_i^{(1)}};....;$$
$$[(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)};C_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,p_i^{(r)}} \tag{1.4}$$

$$\mathbf{B}=[\tau_{i_2}(b_{2ji_2};\beta_{2ji_2}^{(1)},\beta_{2ji_2}^{(2)};B_{2ji_2})]_{1,q_{i_2}},[\tau_{i_3}(b_{3ji_3};\beta_{3ji_3}^{(1)},\beta_{3ji_3}^{(2)},\beta_{3ji_3}^{(3)};B_{3ji_3})]_{1,q_{i_3}};...;$$
$$[\tau_{i_{r-1}}(b_{(r-1)ji_{r-1}};\beta_{(r-1)ji_{r-1}}^{(1)},...,\beta_{(r-1)ji_{r-1}}^{(r-1)};B_{(r-1)ji_{r-1}})]_{1,q_{r-1}} \tag{1.5}$$

$$\mathbf{B}=[\tau_{i_r}(b_{rji_r};\beta_{rji_r}^{(1)},...,\beta_{rji_r}^{(r)};B_{rji_r})]_{1,q_{i_r}} \tag{1.6}$$

$$B=[(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})]_{1,m^{(1)}},[\tau_{i^{(1)}}(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)};D_{ji^{(1)}}^{(1)})]_{m^{(1)}+1,q_i^{(1)}};....;$$
$$[(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})]_{1,m^{(r)}},[\tau_{i^{(r)}}(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)};D_{ji^{(r)}}^{(r)})]_{m^{(r)}+1,q_i^{(r)}} \tag{1.7}$$

$$U=0,n_2;0,n_3;...;0,n_{r-1};\ V=m^{(1)},n^{(1)};m^{(2)},n^{(2)};...;m^{(r)},n^{(r)} \tag{1.8}$$

$$\left.\begin{array}{l} X = p_{i_2}, q_{i_2}, \tau_{i_2}; R_2;...; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}}: R_{r-1}; \\ Y = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}}; R^{(1)};...; p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}}; R^{(r)} \end{array}\right\} \quad (1.9)$$

Let us specify the particular cases of the multivariable Gimel-function by the four conditions :

Condition 1 : If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$. $A_{rj} = A_{rji_r} = B_{rji_r} = 1$, the multivariable Gimel-function reduces to the multivariable Aleph-function defined by Ayant [1].

Condition 2 : If $n_2 = \cdots = n_{r-1} = p_{i_2} = q_{i_2} = \cdots = p_{i_{r-1}} = q_{i_{r-1}} = 0$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the multivariable Gimel-function is replaced by the multivariable I-function defined by Prathima et al. [1].

Condition 3 : If $A_{2j} = A_{2ji_2} = B_{2ji_2} = \cdots = A_{rj} = A_{rji_r} = B_{rji_r} = 1$ and $\tau_{i_2} = \cdots = \tau_{i_r} = \tau_{i^{(1)}} = \cdots = \tau_{i^{(r)}} = R_2 = \cdots = R_r = R^{(1)} = \cdots = R^{(r)} = 1$, then the generalized multivariable Gimel-function is the multivariable I-function defined by Prasad [9].

Condition 4 : If the three above conditions are satisfied at the same time, then the generalized multivariable Gimel-function simplifies to the multivariable H-function.

We will use these conditions in the last section concerning the special cases.

Srivastava ([13], p. 1, Eq. 1) has defined the general class of polynomials

$$S_N^M(x) = \sum_{K=0}^{[N/M]} \frac{(-N)_{MK}}{K!} A_{N,K}\, x^K \quad (1.10)$$

On suitably specializing the coefficients $A_{N,K}$, $S_N^M(x)$ yields a number of known polynomials, these include the Jacobi polynomials, Laguerre polynomials and others polynomials ([17], p. 158-161).

The following serie representation for the $\overline{H}$-function can be obtained from a result given by Rathie ([12], p. 305, Eq. (6.7)), this gives :

$$\overline{H}_{P,Q}^{M,N}\left[z \left| \begin{matrix} (e_j, E_j; \varepsilon_j)_{1,N}, (e_j, E_j)_{N+1,P} \\ (f_j, F_j)_{1,M}, (f_j, F_j; \zeta_j)_{M+1,Q} \end{matrix} \right.\right] = \sum_{t=0}^{\infty} \sum_{h=0}^{M} \overline{\theta}(s_{t,h}) z^{s_{t,h}} \tag{1.11}$$

where

$$\overline{\theta}(s_{t,h}) = \frac{\prod_{j=1, j\neq h}^{M} \Gamma(f_j - F_j s_{t,h}) \prod_{j=1}^{N} [\Gamma(1 - e_j + E_j s_{t,h})]^{\varepsilon_j}}{\prod_{j=M+1}^{Q} [\Gamma(1 - f_j + F_j s_{t,h})]^{\zeta_j} \prod_{j=N+1}^{P} \Gamma(e_j - E_j s_{t,h})} \tag{1.12}$$

The behavior of the $\overline{H}_{P,Q}^{M,N}(z)$ function for small values of z is given by the relations

$$\overline{H}_{P,Q}^{M,N}(z) = 0[|z|^{\alpha}], \text{ where } \alpha = \min_{1\leq j\leq M} \text{Re}\left[\text{Re}\left(\frac{f_j}{F_j}\right)\right], \text{ see B.L.J. Braaksma [3],}$$

and

$$s_{t,h} = \frac{f_h + t}{F_h} \tag{1.13}$$

The (τ, β)-generalized associated Legendre function of first kind is defined and represented in the following form ([18], p. 180, Eq. (15-16))

$$^{\tau,\beta}P_k^{m,n}(z) = \frac{(z+1)^{n/2} (z-1)^{-m/2}}{\Gamma(1-m)}$$

$${}_2F_1^{\tau,\beta}\left(k - \frac{m-n}{2} + 1, -k - \frac{m-n}{2}; 1 - m; \frac{1-z}{2}\right) \tag{1.14}$$

provided $|1 - z| < 2$; $k + \frac{m-n}{2} + 1 \neq 0, -1, -2, ..$; $k - \frac{m-n}{2} \neq 0, \pm 1, \pm 2, ...$; $m \neq 0, -1, -2, ...$ and $|\arg(1 \pm z)| < \pi$; $\{\tau, \beta\} \subset \mathbf{R}$, $\tau > 0$, $\tau - \beta \leq 1$,

where $${}_2F_1^{\tau,\beta}(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1}$$

$${}_2\Psi_1 [(a,1), (c, \tau); (c, \beta); zt^{\tau}]dt \tag{1.15}$$

provided $\{\tau, \beta\} \subset \mathbf{R}$, $\min(\tau, \beta) > 0$, $\tau - \beta \leq 1$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, $\mathrm{Re}(a) > 0$.

The contour representation of (τ, β)-generalized associated Legendre function of first kind which is used in this paper is in the following form

$$
{}^{\tau,\beta}P_k^{m,n}(z) = \frac{(z+1)^{n/2}(z-1)^{-m/2}}{\Gamma\left(k - \frac{m-n}{2} + 1\right)\Gamma\left(-k - \frac{m-n}{2}\right)}
\int_C \frac{\Gamma\left(k - \frac{m-n}{2} + 1 + s\right)\Gamma\left(-k - \frac{m-n}{2} + \tau s\right)\Gamma(-s)}{\Gamma(1 - m + \beta s)} \left(\frac{z-1}{2}\right)^s ds
$$

$$
= \frac{(z+1)^{n/2}(z-1)^{-m/2}}{\Gamma\left(k - \frac{m-n}{2} + 1\right)\Gamma\left(-k - \frac{m-n}{2}\right)}
H_{2,2}^{1,2}\left[\frac{z-1}{2}\,\middle|\,\begin{matrix}\left(-k + \frac{m-n}{2}, 1\right), \left(k + \frac{m-n}{2} + 1, \tau\right)\\ (0, 1), (m, \beta)\end{matrix}\right] \tag{1.16}
$$

2. Required Result :

Lemma :

$$
\int_0^\infty z^{\upsilon-1}(z-1)^{-\mu}(z+1)^{-\rho}(a + bz^q)^{-\sigma}\,dz = \frac{(-)^{\mu}a^{-\sigma}}{\Gamma(\sigma)\Gamma(\mu)}
H_{1,0:1,2;1,2}^{0,1:2,1;1,1}\left[\begin{matrix}\frac{b}{a}\\ -2\end{matrix}\,\middle|\,\begin{matrix}(1-\upsilon; q, 1) : (1-\sigma, 1) ; (1-\mu, 1)\\ -: (0, 1), (\mu - \upsilon + \rho, q) ; (0; 1), (1 - mu - \rho, 1)\end{matrix}\right] \tag{2.1}
$$

Provided $\mathrm{Re}(\upsilon) > 0$, $\mathrm{Re}(\mu) > \mathrm{Re}(\upsilon - \rho) > 0$.

Proof : To establish the above integral, we express the term $(a + bz^q)^{-\rho}$ occurring on its left hand side in terms of Mellin-Barnes contour s-integral ([14], p.18, Eq.(2.1)) and then interchanging the order of s-integral and z-integral. The left hand side of the equation (2.1) takes the following form (say I)

$$I = \frac{a^{-\sigma}}{\Gamma(\sigma)} \frac{1}{2\omega\pi} \int_L \Gamma(\sigma+s)\Gamma(-s)\left(\frac{b}{a}\right)^s \left[\int_0^\infty z^{\upsilon+qs-1}(z-1)^{-\mu}(z+1)^{-\rho}\,dz\right] ds \quad (2.2)$$

Now evaluating the z-integral with the help of the formula ([6], p. 287, Eq. 8), next expressing the Gauss's hypergeometric function involved in the result in terms of contour integral with the help of ([4], p. 62, Eq. 15) and finally interpreting the resulting Mellin-Barnes in terms of H-function of two variables, we obtain the right hand side of (2.1) after algebraic simplifications.

3. Main Integral :

In this section, we determine a generalized infinite integral involving the multivariable Gimel-function with general arguments.

Theorem :

$$\int_0^\infty z^{\upsilon-1}(z-1)^{-\mu}(z+1)^{-\sigma}(a+bz^q)^{-\rho}\, S_{N'}^{M'}\left[y_0 z^{\upsilon_0}(z-1)^{-\mu_0}(z+1)^{\rho_0}(a+bz^q)^{-\sigma_0}\right]$$

$$\bar{H}_{P,Q}^{M,N}\left[y_1 z^{\upsilon^1}(z-1)^{-\mu^1}(z+1)^{-\rho^1}\right] {}^{\tau,\beta}P_k^{m,n}\left(\frac{z}{u}\right) \gimel \begin{pmatrix} z_1 z^{\upsilon_1}(z-1)^{-\mu_1}(z+1)^{\rho_1} \\ \vdots \\ z_1 z^{\upsilon_r}(z-1)^{-\mu_r}(z+1)^{\rho_r} \end{pmatrix} dz$$

$$= \frac{(-)^{-m/2+1}}{\Gamma\left(-k-\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m-n}{2}\right)} \sum_{t=0}^{\infty}\sum_{h=0}^{M}\sum_{K=0}^{[N'/M']} \frac{(-N')_{M'K}}{K!}$$

$$A_{N',K}\, y_0^K\, \bar{\theta}(s_{t,h})\, y_1^{\rho_1 s_{t,h}}\, (-)^{-\mu-\mu_0 K-\mu_1 s_{t,h}}$$

$$\gimel_{X;p_{i_r}+3,q_{i_r}+2,\tau_{i_r}:R_r:Y}^{U;0,n_r+3:V}\left(\begin{matrix} (-)^{-\mu_1}z_1 \\ \cdot \\ \cdot \\ (-)^{-\mu_1}z_1 \\ -2 \\ -2 \end{matrix}\,\middle|\, \begin{matrix} \mathbf{A}; A_1, \mathbf{A} : A; (1,1;1), (1-m,\beta;1) \\ \vdots \\ \vdots \\ \mathbb{B};\mathbf{B}, B_1 : B; (k-\frac{m-n}{2}+1,1;1), (-k-\frac{m-n}{2},\tau;1);(0,1;1) \end{matrix}\right)$$

(3.1)

where

$$A_1 = \left(1 - (\mu - \upsilon + \rho) - \frac{m-n}{2} - (\mu_0 - \upsilon_0 + \rho_0)K - (\mu^1 - \upsilon^1 + \rho^1)s_{t,h}; \mu_1 - \upsilon_1 + \rho_1\right),...,$$

$$(\mu_r - \upsilon_r + \rho_r, 1, 0; 1), \left(1 - \mu - \frac{m}{2} - \mu_0 K - \mu^1_{t,h}, \mu_1,..., \mu_r, 1, 1; 1\right),$$

$$(1 - \upsilon - \upsilon_0 K - \upsilon^1 s_{t,h}, \upsilon_1,..., \upsilon_r, 0, 0; 1) \quad (3.2)$$

$$\mathbf{A} = [(a_{rj}; \alpha^{(1)}_{rj},.., \alpha^{(r)}_{rj}, 0, 0; A_{rj})]_{1,n_r}, [\tau_{i_r}(a_{rji_r}; \alpha^{(1)}_{rji_r},.., \alpha^{(r)}_{rji_r}, 0, 0; A_{rji_r})]_{n+1,p_{i_r}} \quad (3.3)$$

$$\mathbf{B} = [\tau_{i_r}(b_{rji_r}; \beta^{(1)}_{rji_r},..., \beta^{(r)}_{rji_r}, 0, 0; B_{rji_r})]_{1,q_{i_r}} \quad (3.4)$$

$$B_1 = \left(1 - \mu - \rho - \frac{m-n}{2} - (\mu_0 + \rho_0)K - (u^1 + v^1)s_{t,h}; \mu_1 + \rho_1,..., \mu_r + \rho_r, 1, 1:1\right),$$

$$\left(1 - \mu - \frac{m}{2} - \mu_0 K - \mu^1 s_{t,h}; \mu_1,..., \mu_r, 1, 0; 1\right), \quad (3.5)$$

$$U = 0, n_2; 0, n_3;...; 0, n_{r-1}; V = m^{(1)}, n^{(1)}; m^{(2)}, n^{(2)};...; m^{(r)}, n^{(r)}; 2, 1; 1, 0 \quad (3.6)$$

$$X = p_{i_2}, q_{i_2}, \tau_{i_2}; R_2;...; p_{i_{r-1}}, q_{i_{r-1}}, \tau_{i_{r-1}}: R_{r-1} \quad (3.7)$$

$$Y = p_{i(1)}, q_{i(1)}, \tau_{i(1)}; R^{(1)};...; p_{i(r)}, q_{i(r)}, \tau_{i(r)}; R^{(r)}; 2, 2; 0, 1 \quad (3.8)$$

The following quantities $\mathbf{A}$, A, $\mathbf{B}$ and B are defined by the equations (1.2), (1.4), (1.5) and (1.7) respectively in the first section.

Provided that :

$$\upsilon^1, \mu^1, \rho^1, \upsilon_i, \mu_i, \rho_i, \upsilon_i - \mu_i - \rho_i > 0; i = (1,..., r)$$

$$\mathrm{Re}(\upsilon) + \upsilon^1 \min_{1\le j\le M} \mathrm{Re}\left[\left(\frac{f_j}{F_j}\right)\right] + \sum_{i=1}^{r} \upsilon_i \min_{1\le j\le M^{(i)}} \mathrm{Re}\left[D^{(i)}_j\left(\frac{d^{(i)}_j}{\delta^{(i)}_j}\right)\right] > 0$$

$$\mathrm{Re}(\upsilon - \mu - \rho) + (\upsilon^1 - \mu^1 - \rho^1) \min_{1\le j\le M} \mathrm{Re}\left[\left(\frac{f_j}{F_j}\right)\right] + \sum_{i=1}^{r} (\upsilon_i - \mu_i - \rho_i) \min_{1\le j\le M^{(i)}} \mathrm{Re}\left[D^{(i)}_j\left(\frac{d^{(i)}_j}{\delta^{(i)}_j}\right)\right] < 0$$

$|\arg(z_i)| < \frac{1}{2} A^{(k)}_i \pi$, where $A^{(k)}_i$ is defined by Ayant [2].

Proof : To prove the above theorem, expressing the general class of polynomials, the function $\overline{H}(x)$ and (τ, β)-generalized associated Legendre function of first kind in series with the help of (1.6), (1.7) and (1.11) respectively and the multivariable Gimel-function in terms of Mellin-Barnes multiple integrals contour with the help of (1.1). Then interchanging the order of summations and $(s_1,..., s_{r+1})$-integrals with the z-integral which is permissible under the stated conditions. We note I the left hand side of the equation (3.1) and we obtain :

$$I = \frac{(-)^{-\frac{m}{2}} u^{\frac{m-n}{2}}}{\Gamma\left(-k - \frac{m-n}{2} + 1\right)\Gamma\left(k - \frac{m-n}{2}\right)} \sum_{t=0}^{\infty} \sum_{h=0}^{M} \sum_{K=0}^{[N'/M']} \frac{(-N')_{M'K}}{K!}$$

$$A_{N',K}\, y_0^K\, \bar{\theta}(s_{t,h})\, y_1^{s_{t,h}} \frac{1}{(2\pi\omega)^{r+1}}$$

$$\int_{L_1} \cdots \int_{L_{r+1}} \psi(s_1,..., s_r) \sum_{k=1}^{r} \theta_k(s_k) z_k^{s_k}$$

$$\frac{\Gamma\left(k - \frac{m-n}{2} + 1 + s_{r+1}\right)\Gamma\left(-k - \frac{m-n}{2} + \tau s_{r+1}\right)\Gamma(-s_{r+1})}{\Gamma(1 - m + \beta s_{r+1})} \left(\frac{1}{2}\right)^{s_{r+1}}$$

$$\left[\int_0^{\infty} z^{\upsilon+\upsilon K+\upsilon^1 s_{t,h}+\sum_{i=1}^{r}\upsilon_i s_i-1}(z-1)^{-\mu-\frac{m}{2}\mu_0 K-\lambda^1 s_{t,h}-\sum_{i=1}^{r}\mu_i s_i+s_{r+1}}\right.$$

$$\left.(z+1)^{-\rho+\frac{n}{2}\rho_0 K+\mu^1 s_{t,h}-\sum_{i=1}^{r}\rho_i s_i} dz\right] ds_1 .... ds_{r+1} \tag{3.9}$$

Now evaluating the inner integral with the help of the equation (2.1) and expressing the multivariable Gimel-function involved of the result in terms of Mellin-Barnes multiple integrals contour as given by (1.1) and finally reinterpreting the result thus obtained in terms of multivariable Gimel-function of $(r+2)$-variables, we obtain the right hand side of (3.1) after a few simplifications.

4. Special Cases :

In this section, we have several particular cases. We will use the conditions mentioned in the first section.

The multivariable Gimel function reduce to multivariable Aleph-function, we use the conditions 1, and we have the result.

Corollary 1 :

$$\int_0^\infty z^{\upsilon-1}(z-1)^{-\mu}(z+1)^{-\sigma}(a+bz^q)^{-\rho}\, S_{N'}^{M'}\,[y_0 z^{\upsilon_0}(z-1)^{-\mu_0}(z+1)^{\rho_0}(a+bz^q)^{-\sigma_0}]$$

$$\bar{H}_{P,Q}^{M,N}\left[y_1 z^{\upsilon^1}(z-1)^{-\mu^1}(z+1)^{-\rho^1}\right]\,{}^{\tau,\beta}P_k^{m,n}\left(\frac{z}{u}\right)\aleph\begin{pmatrix} z_1 z^{\upsilon_1}(z-1)^{-\mu_1}(z+1)^{\rho_1} \\ \vdots \\ z_r z^{\upsilon_r}(z-1)^{-\mu_r}(z+1)^{\rho_r}\end{pmatrix} dz$$

$$= \frac{(-)^{-m/2+1}}{\Gamma\left(-k-\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m-n}{2}\right)}\sum_{t=0}^{\infty}\sum_{h=0}^{M}\sum_{K=0}^{[N'/M']}\frac{(-N')_{M'K}}{K!}$$

$$A_{N',K}\, y_0^K\,\bar{\theta}(s_{t,h})\, y_1^{\rho_1 s_{t,h}}\,(-)^{-\mu-\mu_0 K-\mu_1 s_{t,h}}$$

$$\aleph_{p_i+3,q_i+2,\tau_i:R:Y}^{0,n+3:V}\left(\begin{matrix}(-)^{-\mu_1}z_1\\ \cdot \\ \cdot \\ (-)^{-\mu_1}z_1 \\ -2 \\ -2\end{matrix}\ \middle|\ \begin{matrix} A_{1,1}, \mathbf{A}_1 : A_1'; (1,1), (1-m,\beta) \\ \vdots \\ \vdots \\ \mathbf{B}_1, B_{1,1} : B_1'; (k-\frac{m-n}{2}+1, 1), (-k-\frac{m-n}{2}, \tau)\,;\,(0,1)\end{matrix}\right) \tag{4.1}$$

under the conditions mentioned in the theorem and we apply the condition 1. In this situation, we have the following relation : $U=X=A=B=0$ and the exponents are equal to 1. The quantities $A_{1,1}, \mathbf{A}_1, A_1, B_{1,1}, \mathbf{B}_1, B_1'$ replace respectively the numbers A_1, $\mathbf{A}$, A, B_1, $\mathbf{B}$, B by respecting the condition 1. The conditions of the theorem and the condition 1 are satisfied. More presicely, we have :

$$A_{1,1} = \left(1-(\mu-\upsilon+\rho)-\frac{m-n}{2}-(\mu_0-\upsilon_0+\rho_0)K-(\mu^1-\upsilon^1+\rho^1)s_{t,h}\,;\,\mu_1-\upsilon_1+\rho_1\right),\ldots,$$

$$(\mu_r-\upsilon_r+\rho_r, 1, 0), \left(1-\mu-\frac{m}{2}-\mu_0 K-\mu^1_{t,h}, \mu_1,\ldots,\mu_r, 1, 1\right),$$

$$(1-\upsilon-\upsilon_0 K-\upsilon^1 s_{t,h}, \upsilon_1,\ldots,\upsilon_r, 0, 0) \tag{4.2}$$

$$B_{1,1} = \left(1 - \mu - \rho - \frac{m-n}{2} - (\mu_0 + \rho_0)K - (u^1 + v^1)s_{t,h}\,;\, \mu_1 + \rho_1, ..., \mu_r + \rho_r, 1, 1\right),$$

$$\left(1 - \mu - \frac{m}{2} - \mu_0 K - \mu^1 s_{t,h};\, \mu_1, ..., \mu_r, 1, 0\right) \quad (4.3)$$

The multivariable Gimel function becomes the multivariable I-function defined by Prathima et al. [11], we use the condition 2, this gives.

Corollary 2 :

$$\int_0^\infty z^{\upsilon-1}(z-1)^{-\mu}(z+1)^{-\sigma}(a+bz^q)^{-\rho}\, S_{N'}^{M'}\,[y_0 z^{\upsilon_0}(z-1)^{-\mu_0}(z+1)^{\rho_0}(a+bz^q)^{-\sigma_0}]$$

$$\overline{H}_{P,Q}^{M,N}\left[y_1 z^{\upsilon^1}(z-1)^{-\mu^1}(z+1)^{-\rho^1}\right]^{\tau,\beta} P_k^{m,n}\left(\frac{z}{u}\right) I \begin{pmatrix} z_1 z^{\upsilon_1}(z-1)^{-\mu_1}(z+1)^{\rho_1} \\ \vdots \\ z_r z^{\upsilon_r}(z-1)^{-\mu_r}(z+1)^{\rho_r} \end{pmatrix} dz$$

$$= \frac{(-)^{-m/2+1}}{\Gamma\left(-k - \frac{m-n}{2} + 1\right)\Gamma\left(k - \frac{m-n}{2}\right)} \sum_{t=0}^{\infty}\sum_{h=0}^{M}\sum_{K=0}^{[N'/M']} \frac{(-N')_{M'K}}{K!}$$

$$A_{N',K}\, y_0^K\, \bar{\theta}(s_{t,h})\, y_1^{\rho_1 s_{t,h}}\, (-)^{-\mu-\mu_0 K-\mu_1 s_{t,h}}$$

$$I_{p+3,q+2:Y_2}^{0,n+3:V} \left(\begin{array}{c|c} (-)^{-\mu_1} z_1 & \mathbf{A}_2, A_1 : A_2; (1, 1; 1), (1-m, \beta; 1) \\ \cdot & \vdots \\ \cdot & \\ (-)^{-\mu_1} z_1 & \vdots \\ -2 & \\ -2 & B_1, \mathbf{B}_2 : B_2 : (k - \frac{m-n}{2} + 1, 1; 1), (-k - \frac{m-n}{2}, \tau; 1)\,;\, (0, 1; 1) \end{array} \right) \quad (4.4)$$

We have the relation : $U = X = A = B = 0$. The numbers $\mathbf{A}_2, A_2, \mathbf{B}_2, B_2, Y_2$ are interchanged by the quantities $\mathbf{A}, A_1, \mathbf{B}, B_1, Y$ respectively by using the condition 2. This corollary verify the conditions concerning the theorem and condition 2,

and

$$Y_2 = p^{(1)}, q^{(1)}; ...; p^{(r)}, q^{(r)}\,;\, 2, 2; 0, 1 \quad (4.5)$$

Now, the multivariable I-function defined by Prasad [9] is a reduction of the multivariable Gimel function. Here, the exponents are equal to 1 and the condition 3 are satisfied, we obtain the relation :

Corollary 3 :

$$\int_0^\infty z^{\upsilon-1}(z-1)^{-\mu}(z+1)^{-\sigma}(a+bz^q)^{-\rho}\, S_{N'}^{M'}[y_0 z^{\upsilon_0}(z-1)^{-\mu_0}(z+1)^{\rho_0}(a+bz^q)^{-\sigma_0}]$$

$$\bar{H}_{P,Q}^{M,N}\left[y_1 z^{\upsilon^1}(z-1)^{-\mu^1}(z+1)^{-\rho^1}\right]{}^{\tau,\beta}P_k^{m,n}\left(\frac{z}{u}\right) I\begin{pmatrix} z_1 z^{\upsilon_1}(z-1)^{-\mu_1}(z+1)^{\rho_1} \\ \vdots \\ z_r z^{\upsilon_r}(z-1)^{-\mu_r}(z+1)^{\rho_r}\end{pmatrix} dz$$

$$=\frac{(-)^{-m/2+1}}{\Gamma\left(-k-\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m-n}{2}\right)}\sum_{t=0}^{\infty}\sum_{h=0}^{M}\sum_{K=0}^{[N'/M']}\frac{(-N')_{M'K}}{K!}$$

$$A_{N',K}\, y_0^K\, \bar{\theta}(s_{t,h})\, y_1^{\rho_1 s_{t,h}}\, (-)^{-\mu-\mu_0 K-\mu_1 s_{t,h}}$$

$$I_{X_3;p_r+3,q_r+2:Y_2}^{U_3;0,n_r+3:V}\left(\begin{matrix} (-)^{-\mu_1}z_1 \\ \cdot \\ \cdot \\ (-)^{-\mu_1}z_1 \\ -2 \\ -2 \end{matrix}\middle| \begin{matrix} \mathbf{A}_3;\mathbf{A}_3, A_{1,1}:A_3;(1,1),(1-m,\beta) \\ \vdots \\ \vdots \\ \mathbf{A}_3;B_{1,1},\mathbf{B}_3:B_3:(k-\frac{m-n}{2}+1,1),(-k-\frac{m-n}{2},\tau);(0,1)\end{matrix}\right) \tag{4.6}$$

In this equation, the exponents are equal to 1. The numbers $\mathbf{A}_3, A_{1,1}, \mathbf{A}_3, A_3$, $\mathbf{B}_3$, $B_{1,1}$, $\mathbf{B}_3$, B_3, Y_2 are instead of the quantities $\mathbf{A}, A_1, \mathbf{A}, A$, $\mathbf{B}, B_1, \mathbf{B}, B$, Y respectively by applying the condition 3. This corollary verify the condition 3 those of the theorem.

Now, the multivariable Gimel function verify the conditions 1, 2, 3 and 4 simultaneously, therefore our function simplify to multivariable H-function, we get the following integral :

Corollary 4 :

$$\int_0^{\infty} z^{\upsilon-1}(z-1)^{-\mu}(z+1)^{-\sigma}(a+bz^q)^{-\rho}\, S_{N'}^{M'}[y_0 z^{\upsilon_0}(z-1)^{-\mu_0}(z+1)^{\rho_0}(a+bz^q)^{-\sigma_0}]$$

$$\bar{H}_{P,Q}^{M,N}\left[y_1 z^{\upsilon^1}(z-1)^{-\mu^1}(z+1)^{-\rho^1}\right]^{\tau,\beta} P_k^{m,n}\left(\frac{z}{u}\right) H\begin{pmatrix} z_1 z^{\upsilon_1}(z-1)^{-\mu_1}(z+1)^{\rho_1} \\ \vdots \\ z_r z^{\upsilon_r}(z-1)^{-\mu_r}(z+1)^{\rho_r} \end{pmatrix} dz$$

$$= \frac{(-)^{-m/2+1}}{\Gamma\left(-k-\frac{m-n}{2}+1\right)\Gamma\left(k-\frac{m-n}{2}\right)} \sum_{t=0}^{\infty}\sum_{h=0}^{M}\sum_{K=0}^{[N'/M']} \frac{(-N')_{M'K}}{K!}$$

$$A_{N',K}\, y_0^K\, \bar{\theta}(s_{t,h})\, y_1^{\rho_1 s_{t,h}}\, (-)^{-\mu-\mu_0 K-\mu_1 s_{t,h}}$$

$$H_{p+3,q+2:Y_2}^{0,n+3:V}\left(\begin{array}{c|c} (-)^{-\mu_1} z_1 & \mathbf{A}_4, A_{1,1} : A_4; (1,1), (1-m,\beta) \\ \cdot & \vdots \\ \cdot & \\ (-)^{-\mu_1} z_1 & \vdots \\ -2 & \\ -2 & B_{1,1}, \mathbf{B}_4 : B_4 : (k-\frac{m-n}{2}+1, 1), (-k-\frac{m-n}{2}, \tau) ; (0,1) \end{array}\right) \quad (4.7)$$

In this above formula, the exponents are equal to 1. The numbers $A_{1,1}$, $\mathbf{A}_4$, A_4, $B_{1,1}$, $\mathbf{B}_4$, B_4, Y_2 are written instead the quantities A_1, $\mathbf{A}$, A, B_1, $\mathbf{B}$, B, Y respectively by verifying conditions 1, 2 and 3 at a time. This integral verify these three conditions cited in the section 1 and the conditions of the theorem.

Remarks :

Jain and Kumawat [8] have obtained easily the same relations with the multivariable H-function. In their document, they gave several applications and specifying certain parameters. This study uses simpler special functions.

We obtain the same relations concerning the multivariable A-function defined by Gautam et al. [5], the modified multivariable H-function defined by Prasad and Singh [10].

5. Conclusion :

The importance of our results lies in their manifold generality. Firstly, in view of the unified infinite integrals with general class of polynomials, the (τ, β)-generalized associated Legendre function of first kind with general arguments utilized in this study, we can obtain a large variety of single simpler infinite integrals specializing the coefficients and the parameters in these functions. Secondly by specializing the various parameters as well as variables in the generalized multivariable Gimel-function, we get a several formulae involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I, Aleph-function of one and several variables and simpler special functions of one and several variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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