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On unified finite integral involving product of Psi-function of two variables and generalized *R*-function

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Abstract:

In this paper, we evaluate a finite integral with the generalized R-function and incomplete Psi-functions of two variables. On account of the most general nature of the functions occurring in the integrand of the second integral, our findings provides interesting unifications and extensions of a number of new and known integrals. At the end of this study, we shall two particular cases.

Keywords: Incomplete Psi-function of two variables, generalized *R*-function, generalized Mittag-Leffler function, incomplete *I*-function of two variables, incomplete *H*-function of two variables, Psi-function of one variable.

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1. Introduction and notations:

First time, we have expressed the incomplete Gamma-functions.

The incomplete Gamma functions $\gamma(\alpha, X)$ and $\Gamma(\alpha, X)$ are defined in the following manner, (see [16]).

$$\gamma(\alpha, X) = \int_0^X u^{\alpha-1} e^{-u} \mathrm{d}u \quad (\Re(\alpha) > 0; X \geqslant 0). \tag{1.1}$$

$$\Gamma(\alpha, X) = \int_X^\infty u^{\alpha-1} e^{-u} \mathrm{d}u \quad (X \geqslant 0; \Re(\alpha) > 0 \text{ when } X = 0). \tag{1.2}$$

We have the following relation :

$$\gamma(\alpha, X) + \Gamma(\alpha, X) = \Gamma(\alpha) \quad (Re(\alpha) > 0) \tag{1.3}$$

Recently Y.P. Kumar and B. Satyanarayana [9] have studied the psi function of one variable. In this paper, we introduce the incomplete psi function of two variables. This function generalizes the I-function of two variables defined by Sharma and Mishra [13], the I-function of two variables introduced by Kumari et al. [5] so the H-function of two variables studied by Gupta and Mittal [3]. It's includes the I-function studied by Saxena [12], the generalized hypergeometric function defined by Rathie [11], the H-function studied by Srivastava et al. [17] and their respective incomplete functions.

Let the Psi function of two variables expressing by the double Mellin Barnes integrals, we have:

$$ {}^{\Gamma}\psi(z_1, z_2 : X) = {}^{\Gamma}\psi^{0,n_1;m_2,n_2:m_3,n_3}_{P_i,Q_i;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''}(z_1, z_2 : X) =$$

$$\left(\begin{array}{c|c} \mathrm{z}_1 & (\mathrm{a}_1, \alpha_1, A_1; \mathbf{A}_1 : X), (a_j, \alpha_j, A_j; \mathbf{A}_j)_{2,n_2}, [(a_{ji}, \alpha_{ji}, A_{ji}; \mathbf{A}_{ji})_{n_2+1,P_i}] \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{z}_2 & (\mathrm{b}_{ji}, \beta_{ji}, B_{ji}; \mathbf{B}_{ji})_{1,Q_i}] : \end{array} \right.$$

$$\left. \begin{array}{c} (\mathrm{c}_j, \gamma_j; \mathbf{C}_j)_{1n_2}, [(c_{ji'}, \gamma_{ji'}; \mathbf{C}_{ji'})_{n_2+1;P_{i'}}]; (e_j, E_j; \mathbf{E}_j)_{1n_3}, [(e_{ji''}, E_{ji''}; \mathbf{E}_{ji''})_{n_3+1;P_{i''}}] \\ \cdot \\ \cdot \\ (\mathrm{d}_j, \delta_j; \mathbf{D}_j)_{1m_2}, [(d_{ji'}, \delta_{ji'}; \mathbf{D}_{ji'})_{m_2+1;Q_{i'}}]; (f_j, F_j; \mathbf{F}_j)_{1m_3}, [(f_{ji''}, F_{ji''}; \mathbf{F}_{ji''})_{m_3+1;Q_{i''}}] \end{array} \right)$$

$$= \frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(t_1,t_2:X)\theta_1(t_1)\theta_2(t_2)z_1^{-t_1}z_2^{-t_2}\mathrm{d}t_1\mathrm{d}t_2 \tag{1.4}$$

where

$$\phi(t_1,t_2:X) = \frac{\Gamma^{\mathbf{A}_1}(1-a_1-\alpha_1 t_1 - A_1 t_2 : X)\prod_{j=2}^{n_1}\Gamma^{\mathbf{A}_j}(1-a_j-\alpha_j t_1 - A_j t_2)}{\sum_{i=1}^{r}\left[\prod_{j=1}^{Q_i}\Gamma^{\mathbf{B}_{ji}}(1-b_{ji}-\beta_{ji}t_1 - B_{ji}t_2)\prod_{n_1+1}^{P_i}\Gamma^{\mathbf{A}_{ji}}\Gamma(a_{ji}+\alpha_{ji}t_1+A_{ji}t_2)\right]} \tag{1.5}$$

$$\theta_1(t_1) = \frac{\prod_{j=1}^{m_2}\Gamma^{\mathbf{D}_j}(d_j+\delta_j t_1)\prod_{j=1}^{n_2}\Gamma^{\mathbf{C}_j}(1-c_j-\gamma_j t_1)}{\sum_{i'=1}^{r'}\left[\prod_{j=n_2+1}^{P_{i'}}\Gamma^{\mathbf{C}_{ji'}}(c_{ji'}+\gamma_{ji'}t_1)\prod_{j=m_2+1}^{Q_{i'}}\Gamma^{\mathbf{D}_{ji'}}(1-d_{ji'}-\delta_{ji'}t_1)\right]} \tag{1.6}$$

$$\theta_2(t_2) = \frac{\prod_{j=1}^{m_3}\Gamma^{\mathbf{F}_j}(f_j+F_j t_2)\prod_{j=1}^{n_3}\Gamma^{\mathbf{E}_j}(1-e_j-E_j t_2)}{\sum_{i''=1}^{r''}\left[\prod_{j=n_3+1}^{P_{i''}}\Gamma^{\mathbf{E}_{ji''}}(e_{ji''}+E_{ji''}t_2)\prod_{j=m_3+1}^{Q_{i''}}\Gamma^{\mathbf{F}_{ji''}}(1-f_{ji''}-F_{ji''}t_2)\right]} \tag{1.7}$$

where z_1 and z_2 (real or complex) are not equal to zero and an empty product is interpreted as unity and the quantities P_i, $P_{i'}$, $P_{i''}$, Q_i, $Q_{i'}$, $Q_{i''}$, m_2, m_3, n_1, n_2, n_3 are non-negative integers such that $Q_i > 0$, $Q_{i'} > 0$, $Q_{i''} > 0 > 0$; $(i = 1,....., r)$, $(i' = 1, ..., r')$, $(i'' = 1, ..., r'')$. The exponents **A**, **B**, **C**, **D**, **E** and **F** are positives numbers.

All the numbers $A's$, $\alpha's$, $B's$, $\beta's$, $\gamma's$, $\delta's$, $E's$ and $F's$ are assumed to be positive quantities for standardization purpose; the definition of Psi-function of two variables given above will however, have a meaning even if some of these quantities are zero and the numbers a_j, a_{ji}, b_{ji}, c_j, d_j, $d_{ji'}$, $c_{ji'}$, f_i, e_i, $f_{ji''}$, $e_{ji''}$ are complex numbers. The contour L_1 is in the s-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma^{C_j}(1 - c_j - \gamma_j t_1)(j = 1,.., n_2)$, $\Gamma^{A_i}(1 - a_j - \alpha_j t_1 - A_j t_2)(j = 1,...., n_1)$ and $\Gamma^{E_j}(1 - e_j - E_j t_2)(j = 1,...., n_3)$ are to the right and all the poles of L_1. The contour L_2 is in the t-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma^{F_j}(f_j + F_j t_2)$, $(j = 1,..., m_3)$ and $\Gamma^{F_j}(d_j + \delta_j t_2)$, $(j = 1,..., m_2)$ lie to the left of L_2. The poles of the integrand are assumed to be simple.

The function defined by the double integrals (1.4) is analytic of z_1 and z_2 if

$$U_1 = \sum_{j=1}^{P_i}\alpha_{ji}\mathbf{A}_{ji} - \sum_{j=1}^{Q_i}\beta_{ji}\mathbf{B}_{ji} + \sum_{j=1}^{P_{i'}}\gamma_{ji'}\mathbf{C}_{ji'} - \sum_{j=1}^{Q_{i'}}\delta_{ji'}\mathbf{D}_{ji'} < 0 \tag{1.8}$$

$$U_2 = \sum_{j=1}^{P_i} A_{ji}\mathbf{A}_{ji} - \sum_{j=1}^{Q_i} B_{ji}\mathbf{B}_{ji} + \sum_{j=1}^{P_{i''}} E_{ji''}\mathbf{E}_{ji''} - \sum_{j=1}^{Q_{i''}} F_{ji''}\mathbf{F}_{ji''} < 0 \tag{1.9}$$

The double integrals defined by (1.4) converges absolutely if

$$U_3 = \sum_{j=1}^{n_1} \alpha_j \mathbf{A}_i + \sum_{j=1}^{n_2} \gamma_j \mathbf{C}_j + \sum_{j=1}^{m_2} \delta_j \mathbf{C}_j - \max_{1\le i\le r}\left(\sum_{j=n_1+1}^{P_i} \alpha_{ji}\mathbf{A}_{ji} + \sum_{j=1}^{Q_i} \beta_{ji}\mathbf{B}_{ji}\right)$$

$$-\max_{1\le i'\le r'}\left(\sum_{j=n_2+1}^{P_{i'}} \gamma_{ji'}\mathbf{C}_{ji'} + \sum_{j=m_2+1}^{Q_{i'}} \delta_{ji'}\mathbf{D}_{ji'}\right) > 0 \tag{1.10}$$

$$U_4 = \sum_{j=1}^{n_1} \alpha_j \mathbf{A}_i + \sum_{i=1}^{m_3} F_j \mathbf{F}_j + \sum_{j=1}^{n_3} E_j \mathbf{E}_j - \max_{1\le i\le r}\left(\sum_{j=n_1+1}^{P_i} A_{ji}\mathbf{A}_{ji} + \sum_{j=1}^{Q_i} B_{ji}\mathbf{B}_{ji}\right)$$

$$-\max_{1\le i''\le r''}\left(\sum_{j=n_3+1}^{P_{i''}} E_{ji''}\mathbf{E}_{ji''} + \sum_{j=m_3+1}^{Q_{i''}} F_{ji''}\mathbf{F}_{ji''}\right) > 0 \tag{1.11}$$

and we have the two inequalities: $|\arg z_1| < \frac{\pi}{2}U_3,\ |\arg z_2| < \frac{\pi}{2}U_4$.

$$\Gamma\psi(z_1, z_2) = 0(|z_1|^{\alpha_1}, |z_r|^{\alpha_2}),\ \max(|z_1|, |z_2|) \to 0$$

$$\Gamma\psi(z_1, z_2) = 0(|z_1|^{\beta_1}, |z_2|^{\beta_2}),\ \min(|z_1|, |z_2|) \to \infty\ :$$

where

$$\alpha_1 = \min_{1\leqslant j\leqslant m_2} Re\left[\left(\mathbf{D}_j \frac{d_j}{\delta_j}\right)\right] \text{ and } \alpha_2 = \min_{1\leqslant j\leqslant m_3} Re\left[\left(\mathbf{F}_j \frac{f_j}{F_j}\right)\right]$$

$$\beta_1 = \max_{1\leqslant j\leqslant n_2} Re\left[\left(\mathbf{C}_j \frac{1-c_j}{\gamma_j}\right)\right] \text{ and } \beta_2 = \max_{1\leqslant j\leqslant n_3} Re\left[\left(\mathbf{E}_j \frac{1-e_j}{E_j}\right)\right],$$

For convenience, we will the notations:

$$A(X) = (a_1, \alpha_1, A_1 : X : \mathbf{A}_1),\ \mathbf{A} = [(a_{ji}, \alpha_{ji}, A_{ji}; \mathbf{A}_{ji})_{n_1+1, P_i}],\ \mathsf{A} = (c_j, \gamma_j, \mathbf{C}_j)_{1, n_2},$$
$$[(c_{ji'}, \gamma_{ji'}; \mathbf{C}_{ji'})_{n_2+1, P_{i'}}],\ [(e_j, E_j; \mathbf{E}_j)_{1, n_3},\ [(e_{ji''}, E_{ji''}; \mathbf{E}_{ji''})_{n_3+1, P_{i''}}] \tag{1.12}$$

$$\mathbf{B} = [(b_{ji}, \beta_{ji}, B_{ji}; \mathbf{B}_{ji})_{1, Q_i}]; \mathsf{B} = (d_j, \delta_j; \mathbf{D}_j)_{1, m_2}, [(d_{ji'}, \delta_{ji'}; \mathbf{D}_{ji'})_{m_2+1, Q_{i'}}],$$
$$(f_j, F_j; \mathbf{F}_i)_{1, m_3}, [(f_{ji''}, F_{ji''}; \mathbf{F}_{ji''})_{m_3+1, Q_{i''}}] \quad (1.13)$$

Now, we give the expression of the gamma incomplete Psi function of two variables.

$$^{\gamma}\psi(z_1, z_2 : X) = {}^{\gamma}\psi^{0,n_1;m_2,n_2;m_3,n_3}_{P_i,Q_i;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''}(z_1, z_2 : X)$$

$$\left(\begin{array}{c|c} z_1 & (a_1, \alpha_1, A_1; \mathbf{A}_1 : X), (a_j, \alpha_j, A_j; \mathbf{A}_j)_{2,n_2}, [(a_{ji}, \alpha_{ji}, A_{ji}; \mathbf{A}_{ji})_{n_2+1,P_i}] : \\ \cdot & \cdot \\ \cdot & \cdot \\ z_2 & (\mathrm{B}_{ji}, \beta_{ji}, B_{ji}; \mathbf{B}_{ji})_{1,Q_i}] : \end{array}\right.$$

$$\left.\begin{array}{c} (c_j, \gamma_j; \mathbf{C}_j)_{1n_2}, [(c_{ji'}, \gamma_{ji'}; \mathbf{C}_{ji'})_{n_2+1;P_{i'}}]; (e_j, E_j; \mathbf{E}_j)_{1n_3}, [(e_{ji''}, E_{ji''}; \mathbf{E}_{ji''})_{n_3+1;P_{i''}}] \\ \cdot \\ \cdot \\ (d_j, \delta_j; \mathbf{D}_j)_{1m_2}, [(d_{ji'}, \delta_{ji'}; \mathbf{D}_{ji'})_{m_2+1;Q_{i'}}]; (f_j, F_j; \mathbf{F}_j)_{1m_3}, [(f_{ji''}, F_{ji''}; \mathbf{F}_{ji''})_{m_3+1;Q_{i''}}] \end{array}\right)$$

$$= \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi'(t_1, t_2 : X) \theta_1(t_1) \theta_2(t_2) z_1^{-t_1} z_2^{-t_2} \mathrm{d}t_1 \mathrm{d}t_2 \quad (1.14)$$

where

$$\phi'(t_1, t_2 : X) = \frac{\gamma^{\mathbf{A}_1}(1 - a_1 - \alpha_1 t_1 - A_1 t_2 : X) \prod_{j=2}^{n_1} \Gamma^{\mathbf{A}_j}(1 - a_j - \alpha_j t_1 - A_j t_2)}{\sum_{i=1}^{r} \left[\prod_{j=1}^{Q_i} \Gamma^{\mathbf{B}_{ji}}(1 - b_{ji} - \beta_{ji} t_1 - B_{ji} t_2) \prod_{n_1+1}^{P_i} \Gamma^{\mathbf{A}_{ji}} \Gamma(a_{ji} + \alpha_{ji} t_1 + A_{ji} t_2)\right]} \quad (1.15)$$

$\theta_1(s)$ and $\theta_2(t)$ are defined by the equation (1.6) and (1.7) respectively. We have the equality

$$^{\Gamma}\psi(z_1, z_2 : X) + ^{\gamma}\psi(z_1, z_2 : X) = \psi(z_1, z_2) \quad (1.16)$$

In the right side of the above equation, we get the Psi function of two variables.

Recently Kumar and Kumar [4] have defined the generalized R-function as follows:

$$^{k}_{p}R^{\alpha,\beta,\gamma}_{q}(z) = {}^{k}_{p}R^{\alpha,\beta,\gamma}_{q}(a_1, \cdots, a_p; b_1, \cdots, b_q; z) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{(\gamma)_{kn} z^n}{\Gamma(\alpha n+\beta) n!} \quad (1.17)$$

We note

$$^{k}_{p}A^{\alpha,\beta,\gamma}_{q}(n) = \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{(\gamma)_{kn}}{\Gamma(\alpha n+\beta) n!} \quad (1.18)$$

where $\alpha, \beta, \gamma \in C$, $Re(\alpha) > \max\{0, Re(k) - 1\}$, $Re(k) > 0$. About the absolutely and normal convergent for instance, see [4].

Let $a_j(j = 1,...., p) = b_j(j = 1,...., q) = 1$, the above function reduces to Generalized Mittag-Leffler function defined by Srivastava and Tomovski [18], this gives:

$$\mathrm{E}^{\gamma,k}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{kn} z^n}{\Gamma(\alpha n+\beta)n!} \tag{1.19}$$

where $\alpha, \beta, \gamma, k, z \in C$, $\min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, $Re(\alpha) > \max\{0, R(k) - 1\}$.

We note:

$$\mathrm{A}^{\gamma,k}_{\alpha,\beta}(n) = \frac{(\gamma)_{kn}}{\Gamma(\alpha n+\beta)p!} \tag{1.20}$$

Taking $q = k$, we obtain the generalized Mittag-Leffler function defined by Shukla and Prajapati [14].

$$\mathrm{E}^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} z^n}{\Gamma(\alpha n+\beta)n!} \tag{1.21}$$

where $\min\{Re(\beta), Re(\gamma)\} > 0$,we note

$$\mathrm{A}^{\gamma,q}_{\alpha,\beta}(n) = \frac{(\gamma)_{qn}}{\Gamma(\alpha n+\beta)n!} \tag{1.22}$$

We suppose : $a_j(j = 1,...., p) = b_j(j = 1,...., q) = k = 1$, we obtain the function defined by Prabhakar [8]:

$$\mathrm{E}^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n+\beta)n!} \tag{1.23}$$

under the conditions where $z, \alpha, \beta, \gamma \in C$, $\min\{Re(\alpha), Re(\beta)\} > 0$. We pose:

$$\mathrm{A}^{\gamma}_{\alpha,\beta}(n) = \frac{(\gamma)_n}{\Gamma(\alpha n+\beta)n!} \tag{1.24}$$

If $\gamma = 1$, we obtain the Mittag-Leffler function defined by Wiman [19, 20] and we have:

$$\mathrm{E}_{\alpha,\beta}(z) = \sum_{n=1}^{\infty} \frac{z^p}{\Gamma(\alpha z+\beta)n!} \tag{1.25}$$

where $z, \alpha, \beta \in C$, $\min\{Re(\alpha), Re(\beta)\} > 0$.

Let

$$\mathrm{A}_{\alpha,\beta}(n) = \frac{1}{\Gamma(\alpha n+\beta)n!} \tag{1.26}$$

Now, β = 1, we get the Mittag-Leffler function [6, 7]:

$$\mathrm{E}_{\beta}(z) = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)n!} \tag{1.27}$$

where $z, \alpha \in C$.

Let

$$\mathrm{A}_{\alpha}(n) = \frac{1}{\Gamma(\alpha n+1)n!} \tag{1.28}$$

2. Required integral:

We have the following integral, see Prudnikov et al. ([10], 4.1.5, 61 page 139).

Lemma:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln \left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] \mathrm{d}x = 2a^{s+t} B\left(s+\frac{1}{2}, t+\frac{1}{2}\right)$$

$$\times {}_4F_3\left(\begin{array}{c} \frac{1}{2}, 1, s+\frac{1}{2}, t+\frac{1}{2} \\ \cdots \\ \frac{3}{2}, \frac{s+t}{2}+1, \frac{s+t+1}{2} \end{array}; \frac{(ab)^2}{4}\right) \tag{2.1}$$

where $a > -1$, $Re(s) > -1$, $Re(t) > -1$, $|\arg(4 - a^2b^2)| < \pi$.

In the following section, we give an unified finite integral concerning the incomplete multivariable Psi-function this gives :

3. Main integral:

Let

$$V = m_2, n_2; m_3, n_3; Y = P_{i'}, Q_{i'}; r'; P_{i''}, Q_{i''}; r'' \tag{3.1}$$

We have the following result by using the notations given by (1.12) and (1.13):

Theorem 1:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln \left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] {}_p^{k'}R_q^{\alpha,\beta,\gamma}(zx^A(1-x)^B)$$

$$ {}^{(\Gamma)}\psi\begin{pmatrix} Z_1 x^{\eta_1}(a-x)^{\epsilon_1} \\ \cdot \\ \cdot \\ Z_2 x^{\eta_2},(a-x)^{\epsilon_2} \end{pmatrix} dx = 2a^{s+t}\sum_{n,n'=0}^{\infty} {}_p^{k'}A_q^{\alpha,\beta,\gamma}(n)\, z^n \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}} \left(\frac{(ab)^2}{4}\right)^{n'} $$

$$ {}^{(\Gamma)}\psi^{0,n_1+4:V}_{P_i+4,Q_i+3;r:Y}\left(\begin{array}{c|c} Z_1 & \mathrm{A(X)}, \mathrm{A}_1, \mathbf{A} : \mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_2 & \mathbf{B}, \mathrm{B}_1 : \mathfrak{B} \end{array}\right) \tag{3.2} $$

under the conditions verified by the psi-function of two variables, see the first section and

$$ Re\,(s+nA)+(\eta_1+\eta_2)\min_{1\leqslant j\leqslant m_2} Re\left[\left(\mathbf{D}_j\frac{d_j}{\delta_j}\right)\right] > -1 $$

and

$$ Re\,(t+nB)+(\epsilon_1+\epsilon_2)\min_{1\leqslant j\leqslant m_3} Re\left[\left(\mathbf{F}_j\frac{f_j}{F_j}\right)\right] > -1 $$

where $\alpha, \beta, \gamma \in C$, $Re(\alpha) > \max\{0, Re(k') - 1\}$, $Re(k') > 0$. A, B > 0 and $|\arg(z_k)| < \frac{1}{2}A^{(k)}\pi$, $A^{(k)}$ is defined by (1.10) and (1.11), $k=1,2\ldots.$

and

$$ \mathrm{A}_1 = \left(\tfrac{1}{2}-s-n'-nA;\eta_1,\eta_2;1\right),\left(\tfrac{1}{2}-t-n'-nB;\epsilon_1,\epsilon_2;1\right), $$

$$ \left(\tfrac{-s-t-n(A+B)}{2};\tfrac{\eta_1+\epsilon_1}{2},\tfrac{\eta_2+\epsilon_2}{2};1\right),\left(\tfrac{1-s-t-n(A+B)}{2};\tfrac{\eta_1+\epsilon_1}{2},\tfrac{\eta+\epsilon}{2};1\right) \tag{3.3} $$

$$ \mathrm{B}_1 = \left(\tfrac{-s-t-n(A+B)}{2}-n';\tfrac{\eta_1+\epsilon_1}{2},\tfrac{\eta_2+\epsilon_2}{2};1\right),\left(\tfrac{1-s-t-n(A+B)}{2}-n';\tfrac{\eta_1+\epsilon_1}{2},\tfrac{\eta_2+\epsilon_2}{2};1\right), $$

$$ (-s-t-n(A+B);\eta_1+\epsilon_1,\eta_2+\epsilon_2;1) \tag{3.4} $$

Proof: To prove the theorem, first expressing the new generalization R-function defined by [4] in series with the help of (1.17), and we interchange the order of summations and integral (which is permissible under the conditions stated). Expressing the incomplete Psi-function of two variables in double Mellin-Barnes contour integrals with the help of (1.7) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process, we will note the left hand side of (3.2) I:

$$I = \sum_{n=0}^{\infty} {}_{A}^{k'}R_{q}^{\alpha,\beta,\gamma}(zx^{A}(1-x)^{B})\, z^{n}\ \frac{1}{(2\pi\omega)^{2}} \int_{L_1}\int_{L_2} {}^{(\Gamma)}\phi(t_1,t_2;X)\ \prod_{k=1}^{2}\theta_k(s_k) Z_k^{s_k}$$

$$\int_0^a x^{s+nA+\sum_{i=1}^{2} t_i\eta_i}(a-x)^{t+nB+\sum_{i=1}^{2} t_k\epsilon_k}\, dx dt_1 dt_2 \tag{3.5}$$

Using the lemma, we obtain:

$$I = \sum_{n=0}^{\infty} {}_{p}^{k'}A_{q}^{\alpha,\beta,\gamma}(n) z^{n}\ \frac{1}{(2\pi\omega)^{2}} \int_{L_1}\int_{L_2} {}^{(\Gamma)}\phi(t_1,t_2;X) \prod_{k=1}^{2}\theta_k(t_k) Z_k^{s_k}$$

$$B\left(s+nA+\sum_{k=1}^{2}\eta_k t_k+\frac{1}{2}, t+nB+\sum_{k=1}^{2}\epsilon_k t_k+\frac{1}{2}\right)$$

$${}_4F_3\left(\begin{array}{c} \frac{1}{2}, 1, s+nA+\sum_{k=1}^{2}\eta_k t_k+\frac{1}{2}, t+nB+\sum_{k=1}^{2}\epsilon_k t_k+\frac{1}{2} \\ \cdots \\ \frac{3}{2}, \frac{s+t+n(A+B)+\sum_{k=1}^{2} t_k(\eta_k+\epsilon_k)}{2}+1, \frac{s+t+n(A+B)+\sum_{k=1}^{2} t_k(\eta_k+\epsilon_k)+1}{2} \end{array}; \frac{(ab)^2}{4}\right) dt_1 dt_2 \tag{3.6}$$

Now, replacing the Gauss hypergeometric function by the serie $\sum_{n'=0}^{\infty}$ (see [15]), under the hypothesis, we can interchanged this serie, the (t_1, t_2)-integrals and applying the definition of Beta-function, we have:

$$I = 2a^{s+t} \sum_{n,n'=0}^{\infty} {}_{p}^{k'}A_{q}^{\alpha,\beta,\gamma}(n) z^{n}\ \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}} \left(\frac{(ab)^2}{4}\right)^{n'} \frac{1}{(2\pi\omega)^{2}} \int_{L_1}\int_{L_2}$$

$${}^{(\Gamma)}\phi(t_1,t_2;X) \prod_{k=1}^{2}\theta_k(t_k)\ Z_k^{s_k} \frac{\Gamma\left(s+na+\sum_{k=1}^{2}\eta_k t_k+\frac{1}{2}\right)\Gamma\left(t+nb+\sum_{k=1}^{2}\epsilon_k t_k+\frac{1}{2}\right)}{\Gamma\left(s+t+n(a+b)+\sum_{k=1}^{2}(\eta_k+\epsilon_k)t_k+1\right)}$$

$$\frac{\left(s+na+\sum_{k=1}^{2} t_k\eta_k+\frac{1}{2}\right)_{n'}\left(t+nb+\sum_{k=1}^{2} t_k\epsilon_k+\frac{1}{2}\right)_{n'}}{\left(\frac{s+t+n(a+b)+\sum_{k=1}^{2} t_k(\eta_k+\epsilon_k)}{2}+1\right)_{n'}\left(\frac{s+t+n(a+b)+\sum_{k=1}^{2} t_k(\eta_k+\epsilon_k)+1}{2}\right)_{n'}} dt_1 dt_2 \tag{3.7}$$

Applying the relations $(a)_{n'} = \dfrac{\Gamma(a+n')}{\Gamma(a)}$ and $\Gamma(a)(a)_{n'} = \Gamma(a+n')$ this gives

$$I = 2a^{s+t}\ {}_{p}^{k'}A_{q}^{\alpha,\beta,\gamma}(n) z^{n}\ \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}} \left(\frac{(ab)^2}{4}\right)^{n'} \frac{1}{(2\pi\omega)^{2}} \int_{L_1}\int_{L_2} {}^{(\Gamma)}\phi(t_1,t_2;X) \prod_{k=1}^{2} Z_k^{s_k}\ \theta_k(t_k)$$

$$\frac{\Gamma\left(s+na+\sum_{k=1}^{2}\eta_k t_k+n'+\frac{1}{2}\right)\Gamma\left(t+nb+\sum_{k=1}^{2}\epsilon_k t_k+n'+\frac{1}{2}\right)}{\Gamma\left(s+t+n(a+b)+\sum_{k=1}^{2}(\eta_k+\epsilon_k)t_k+1\right)}$$

$$\frac{\Gamma\left(\frac{s+t+n(a+b)+\sum_{k=1}^{2}t_k(\eta_k+\epsilon_k)t_k}{2}+1\right)}{\Gamma\left(\frac{s+t+n(a+b)+\sum_{k=1}^{2}t_k(\eta_k+\epsilon_k)t_k}{2}+1+n'\right)}$$

$$\frac{\Gamma\left(\frac{s+t+n(a+b)+\sum_{k=1}^{2}t_k(\eta_k+\epsilon_k+1)t_k}{2}\right)}{\Gamma\left(\frac{s+t+n(a+b)+\sum_{k=1}^{2}t_k(\eta_k+\epsilon_k)t_k+1}{2}+n'\right)}\,\mathrm{d}t_1\mathrm{d}t_2 \tag{3.8}$$

Interpreting the equation (3.8) to incomplete Psi function of two variables, we get the result (3.2).

Consider the gamma incomplete Psi-function of two variables, we have:

Theorem 2:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln\left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] {}_p^{k'}R_q^{\alpha,\beta,\gamma}(zx^A(1-x)^B)$$

$$^{(\gamma)}\psi\begin{pmatrix} Z_1x^{\eta_1}(a-x)^{\epsilon_1} \\ \cdot \\ \cdot \\ Z_2x^{\eta_2},(a-x)^{\epsilon_2}\end{pmatrix}\mathrm{d}x = 2a^{s+t}\sum_{n,n'=0}^{\infty} {}_p^{k'}A_q^{\alpha,\beta,\gamma}(n)z^n\,\frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}}\left(\frac{(ab)^2}{4}\right)^{n'}$$

$$^{(\gamma)}\psi^{0,n_1+4:V}_{P_i+4,Q_i+3;r:Y}\left(\begin{array}{c|c} Z_1 & \mathrm{A(X)},\,\mathrm{A}_1,\mathbf{A}:\mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_2 & \mathbf{B},\,\mathrm{B}_1:\mathfrak{B}\end{array}\right) \tag{3.9}$$

In the section 4, we will see the case $r = 1$, we consider the different generalizations of the Mittag-Leffler functions cited in the sections 1.

4. Special cases:

In the following, we will consider the Gamma incomplete Psi function of two variables. We consider the generalized Mittag-Leffler function defined by [18], this gives:

Corollary 1:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln\left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] \mathrm{E}_{\alpha,\beta}^{\gamma,k'}(zx^A(a-x)^B)\ {}^{(\Gamma)}\psi\begin{pmatrix} Z_1 x^{\eta_1}(a-x)^{\epsilon_1} \\ \cdot \\ \cdot \\ Z_2 x^{\eta_2},(a-x)^{\epsilon_2}\end{pmatrix}$$

$$dx = 2a^{s+t}\sum_{n,n'=0}^{\infty} \mathrm{A}_{\alpha,\beta}^{\gamma,k'}(n)z^n\ \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}}\left(\frac{(ab)^2}{4}\right)^{n'}{}^{(\Gamma)}\psi_{P_i+4,Q_i+3;r:Y}^{0,n_1+4:V}\left(\begin{array}{c|c} Z_1 & \mathrm{A(X)}, \mathrm{A}_1, \mathbf{A}:\mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_2 & \mathbf{B}, \mathrm{B}_1:\mathfrak{B}\end{array}\right) \tag{4.1}$$

under the conditions and notations verified by the theorem and where $\alpha, \beta, \gamma, k, z \in C$, $\min\{Re(\alpha), Re(\beta), Re(\gamma)\} > 0$, $Re(\alpha) > \max\{0, R(k) - 1\}$.

Taking the generalized Mittag-Leffler function defined by Shukla and Prajapati [14], this gives:

Corollary 2:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln\left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] \mathrm{E}_{\alpha,\beta}^{\gamma,q}(zx^A(a-x)^B)\ {}^{(\Gamma)}\psi\begin{pmatrix} Z_1 x^{\eta_1}(a-x)^{\epsilon_1} \\ \cdot \\ \cdot \\ Z_2 x^{\eta_2},(a-x)^{\epsilon_2}\end{pmatrix}$$

$$dx = 2a^{s+t}\sum_{n,n'=0}^{\infty} \mathrm{A}_{\alpha,\beta}^{\gamma,q}(n)z^n\ \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}}\left(\frac{(ab)^2}{4}\right)^{n'}{}^{(\Gamma)}\psi_{P_i+4,Q_i+3;r:Y}^{0,n_1+4:V}\left(\begin{array}{c|c} Z_1 & \mathrm{A(X)}, \mathrm{A}_1, \mathbf{A}:\mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_2 & \mathbf{B}, \mathrm{B}_1:\mathfrak{B}\end{array}\right) \tag{4.2}$$

under the conditions and notations verified by the theorem and $\alpha, \beta, \gamma, z \in C$, min $\{Re(\beta), Re(\gamma)\} > 0$.

Corollary 3:

Let the generalized Mittag-Leffler function defined by Prabhakar [8], we get:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln\left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] \mathrm{E}^{\gamma}_{\alpha,\beta}(zx^A(a-x)^B)\ {}^{(\Gamma)}\psi\begin{pmatrix} Z_1 x^{\eta_1}(a-x)^{\epsilon_1} \\ \cdot \\ \cdot \\ Z_2 x^{\eta_2},(a-x)^{\epsilon_2}\end{pmatrix}$$

$$\mathrm{d}x = 2a^{s+t}\sum_{n,n'=0}^{\infty} \mathrm{A}^{\gamma}_{\alpha,\beta}(n) z^n\ \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}}\left(\frac{(ab)^2}{4}\right)^{n'} {}^{(\Gamma)}\psi^{0,n_1+4:V}_{P_i+4,Q_i+3;r:Y}\left(\begin{array}{c|c} Z_1 & \mathrm{A(X)},\ \mathrm{A}_1, \mathbf{A}:\mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_2 & \mathbf{B},\ \mathrm{B}_1:\mathfrak{B}\end{array}\right) \tag{4.3}$$

under the conditions and notations verified by the theorem and $z, \alpha, \beta, \gamma \in C$, min $\{Re(\alpha), Re(\beta)\} > 0$.

Corollary 4:

Considering the generalized Mittag-Leffler function defined by Wiman [19, 20], we get:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln\left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] \mathrm{E}_{\alpha,\beta}(zx^A(a-x)^B)\ {}^{(\Gamma)}\psi\begin{pmatrix} Z_1 x^{\eta_1}(a-x)^{\epsilon_1} \\ \cdot \\ \cdot \\ Z_2 x^{\eta_2},(a-x)^{\epsilon_2}\end{pmatrix}$$

$$\mathrm{d}x = 2a^{s+t}\sum_{n,n'=0}^{\infty} \mathrm{A}_{\alpha,\beta}(n) z^n\ \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}}\left(\frac{(ab)^2}{4}\right)^{n'} {}^{(\Gamma)}\psi^{0,n_1+4:V}_{P_i+4,Q_i+3;r:Y}\left(\begin{array}{c|c} Z_1 & \mathrm{A(X)},\ \mathrm{A}_1, \mathbf{A}:\mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ Z_2 & \mathbf{B},\ \mathrm{B}_1:\mathfrak{B}\end{array}\right) \tag{4.4}$$

under the conditions and notations verified by the theorem and $z, \alpha, \beta \in C$, min $\{Re(\alpha), Re(\beta)\} > 0$.

Corollary 5:

Now, we consider the generalized Mittag-Leffler function defined by [6, 7], we get:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln\left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] \mathrm{E}_\alpha(zx^A(a-x)^B)\ {}^{(\Gamma)}\psi\left(\begin{array}{c} \mathrm{Z}_1 x^{\eta_1}(a-x)^{\epsilon_1} \\ \cdot \\ \cdot \\ \mathrm{Z}_2 x^{\eta_2}, (a-x)^{\epsilon_2} \end{array}\right)$$

$$\mathrm{d}x = 2a^{s+t} \sum_{n,n'=0}^{\infty} \mathrm{A}_\alpha(n)\mathrm{z}^n \ \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}} \left(\frac{(ab)^2}{4}\right)^{n'} {}^{(\Gamma)}\psi^{0,n_1+4:V}_{P_i+4,Q_i+3;r:Y}\left(\begin{array}{c|c} \mathrm{Z}_1 & \mathrm{A(X)}, \mathrm{A}_1, \mathbf{A} : \mathfrak{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \mathrm{Z}_2 & \mathbf{B}, \mathrm{B}_1 : \mathfrak{B} \end{array}\right) \tag{4.5}$$

under the conditions and notations verified by the theorem and z, $\alpha \in C$, min $\{Re(\alpha)\} > 0$.

Remark: We obtain the same results with the gamma incomplete Psi function of two variables.

Now, we take the generalized R-function and the Psi-function of one variable [9], this gives:

Corollary 6:

$$\int_0^a x^{s-1}(a-x)^{t-1} ln\left[\frac{1+b\sqrt{x(a-x)}}{1-b\sqrt{x(a-x)}}\right] {}^{k'}_{p}R^{\alpha,\beta,\gamma}_{q}(zx^A(1-x)^B)$$

$$^{(\Gamma)}\psi(\ \mathrm{Z}\mathrm{x}^{\eta}(a-x)^{\epsilon}\ )\mathrm{d}x = 2a^{s+t} \sum_{n,n'=0}^{\infty} {}^{k'}_{p}A^{\alpha,\beta,\gamma}_{q}(n)\mathrm{z}^n \ \frac{\left(\frac{1}{2}\right)_{n'}}{\left(\frac{3}{2}\right)_{n'}} \left(\frac{(ab)^2}{4}\right)^{n'}$$

$$^{(\Gamma)}\psi^{0,n_1+4}_{P_i+4,Q_i+3;r}\left(\begin{array}{c|c} & \mathrm{A}_{11}(X), A_{11}, \mathbf{A}_1 \\ & \cdot \\ \mathrm{Z} & \cdot \\ & \cdot \\ & \mathbf{B}_1, \mathrm{B}_{11} \end{array}\right) \tag{4.6}$$

where

$$\mathrm{A}_{11} = \left(\tfrac{1}{2} - s - n' - nA; \eta; 1\right), \left(\tfrac{1}{2} - t - n' - nB; \epsilon; 1\right), \left(\tfrac{-s-t-n(A+B)}{2}; \tfrac{\eta+\epsilon}{2}; 1\right),$$

$$\left(\tfrac{1-s-t-n(A+B)}{2}; \tfrac{\eta+\epsilon}{2}; 1\right) \tag{4.7}$$

$$\mathrm{B}_{11} = \left(\frac{-s-t-n(A+B)}{2} - n'; \frac{\eta+\epsilon}{2}; 1\right), \left(\frac{1-s-t-n(A+B)}{2} - n'; \frac{\eta+\epsilon}{2}; 1\right),$$

$$(-s-t-n(A+B); \eta+\epsilon; 1) \tag{4.8}$$

$$A_{11}(X) = [(\mathrm{a}_1; \alpha_1^{(1)}, X; A_1)], \mathbf{A}_1 = [(a_j; \alpha_j; A_j)]_{2,n}, [(\mathrm{a}_{ji}; \alpha_{ji}; A_{ji}]_{n+1,p_i} \tag{4.9}$$

$$\mathbf{B}_1 = [(b_j; \beta_j; B_j)]_{1,m}, \quad [(\mathrm{b}_j; \beta_{ji}; B_{ji})]_{m+1,q_i} \tag{4.10}$$

Remarks: We obtain the similar finite integrals with the incomplete I-function [1].

If the exponents are equals to 1, the incomplete psi function of two variables reduces to incomplete I-function of two variables [13] and we have the same formulas.

If $r = r' = r'' = 1$, we obtain the incomplete of I-function of two variables [5] and we obtain the similar results.

If the two aboves conditions are satisfied at the time, we have the incomplete of the H-function of two variables [3].

If $X = 0$, we obtain the same integrals about the psi function of two variables, the I-function of two variables [13], the I-function of two variables [5] and the H-function of two variables [3].

5. Conclusion:

In this paper, we have an unified finite integral. The importance of our all the results lies in their manifold generality. By specializing the parameters as well as variables in the multivariable Psi-function. We can find a big number of formulas involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I functions and simpler special functions of one and two variables. Secondly by varying the parameters and variables of the new generalized R-function defined by Kumar and Kumar [4] involved here, we can obtain a large number of special functions of one variable and new formulas. Thirdly, concerning this finite integral, we can have a large number of finite integrals about the special functions of several or one variables. Hence the results derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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