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Integral Involving the Generalized Psi Function of Two Variables

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Abstract :

Goyal and Agrawal [5] have studied the MacRobert's integral involving the I-function of two variables. In the present document, we introduced the generalized Psi function of two variables and we calculate the MacRobert's integral with the generalized Psi-function of two variables. At the end, we will see several particular cases and remarks.

Keywords : Generalized Psi-function of two variables, Psi-function of two variables, *I*-function of two variables, *H*-function of two variables.

1. Introduction and notations :

Recently Y.P. Kumar and B. Satyanarayana [8] have studied the Psi-function of one variable. In this present paper, we introduce the generalized Psi-function of two variables. This function is an extension of Psi-function of two variables, consequently the I-function of two variables defined by Sharma and Mishra [16], the I-function of two variables studied by Kumari et al. [9] so the H-function of two variables studied by Gupta and Mittal [6]. It's includes the I-function studied by Saxena [13], the generalized hypergeometric function introduced by Rathie [12].

The double Mellin-Barnes integrals contour occurring in this paper will be refered to as the generalized Psi-function of two variables throughout our present study and will be defined and represented as follows : Let the generalized Psi function of two variables, we have :

$$\psi(z_1,z_2)=\psi^{m_1,n_1;m_2,n_2;m_3,n_3}_{P_i,Q_i;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''}\left(\begin{array}{c|l} \mathrm{z}_1 & (\mathrm{a}_j,\alpha_j,A_j;\mathbf{A}_j)_{1,n_1},[(a_{ji},\alpha_{ji},A_{ji};\mathbf{A}_{ji})_{n_1+1,P_i}]: \\ \cdot & \qquad\qquad\qquad\cdot \\ \cdot & \qquad\qquad\qquad\cdot \\ \mathrm{z}_2 & (\mathrm{b}_j,\beta_j,B_j;\mathbf{B}_j)_{1,m_1},[(b_{ji},\beta_{ji},B_{ji};\mathbf{B}_{ji})_{m_1+1,Q_i}]: \end{array}\right.$$

$$\left.\begin{array}{l} (\mathrm{c}_j,\gamma_j;\mathbf{C}_j)_{1n_2},[(c_{ji'},\gamma_{ji'};\mathbf{C}_{ji'})_{n_2+1;P_{i'}}];(e_j,E_j;\mathbf{E}_j)_{1n_3},[(e_{ji''},E_{ji''};\mathbf{E}_{ji''})_{n_3+1;P_{i''}}] \\ \qquad\qquad\qquad\qquad\cdot \\ \qquad\qquad\qquad\qquad\cdot \\ (\mathrm{d}_j,\delta_j;\mathbf{D}_j)_{1m_2},[(d_{ji'},\delta_{ji'};\mathbf{D}_{ji'})_{m_2+1;Q_{i'}}];(f_j,F_j;\mathbf{F}_j)_{1m_3},[(f_{ji''},F_{ji''};\mathbf{F}_{ji''})_{m_3+1;Q_{i''}}] \end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,t)\theta_1(s)\theta_2(t)z_1^{-s}z_2^{-t}\mathrm{d}s\mathrm{d}t \tag{1.1}$$

where

$$\phi(s,t)=\frac{\prod_{j=1}^{n_1}\Gamma^{\mathbf{A}_j}(1-a_j-\alpha_j s-A_j t)\prod_{j=1}^{m_1}\Gamma^{\mathbf{B}_j}(b_j+\beta_j s+B_j t)}{\sum_{i=1}^{r}\left[\prod_{j=m_1+1}^{Q_i}\Gamma^{\mathbf{B}_{ji}}(1-b_{ji}-\beta_{ji}s-B_{ji}t)\prod_{n_1+1}^{P_i}\Gamma^{\mathbf{A}_{ji}}\Gamma(a_{ji}+\alpha_{ji}s+A_{ji}t)\right]} \tag{1.2}$$

$$\theta_1(s)=\frac{\prod_{j=1}^{m_2}\Gamma^{\mathbf{D}_j}(d_j+\delta_j s)\prod_{j=1}^{n_2}\Gamma^{\mathbf{C}_j}(1-c_j-\gamma_j s)}{\sum_{i'=1}^{r'}\left[\prod_{j=n_2+1}^{P_{i'}}\Gamma^{\mathbf{C}_{ji'}}(c_{ji'}+\gamma_{ji'}s)\prod_{j=m_2+1}^{Q_{i'}}\Gamma^{\mathbf{D}_{ji'}}(1-d_{ji'}-\delta_{ji'}s)\right]} \tag{1.3}$$

$$\theta_2(t)=\frac{\prod_{j=1}^{m_3}\Gamma^{\mathbf{F}_j}(f_j+F_j t)\prod_{j=1}^{n_3}\Gamma^{\mathbf{E}_j}(1-e_j-E_j t)}{\sum_{i''=1}^{r''}\left[\prod_{j=n_3+1}^{P_{i''}}\Gamma^{\mathbf{E}_{ji''}}(e_{ji''}+E_{ji''}t)\prod_{j=m_3+1}^{Q_{i''}}\Gamma^{\mathbf{F}_{ji''}}(1-f_{ji''}-F_{ji''}t)\right]} \tag{1.4}$$

where z_1 and z_2 (real or complex) are not equal to zero and an empty product is interpreted as unity and the quantities $P_i, P_{i'}, P_{i''}, Q_i, Q_{i'}, Q_{i''}, m_1, m_2, m_3, n_1, n_2, n_3$ are non-negative integers such that $Q_i > 0$, $Q_{i'} > 0$, $Q_{i''} > 0$; $(i = 1,....., r)$, $(i' = 1, ..., r')$, $(i'' = 1, ..., r'')$. The exponents A, B, C, D, E and F are positives numbers.

All the numbers $A's$, $\alpha's$, $B's$, $\beta's$, $\gamma's$, $\delta's$, $E's$ and $F's$ are assumed to be positive quantities for standardization purpose; the definition of Psi-function of two variables given above will however, have a meaning even if some of these quantities are zero and the numbers a_j, b_j, a_{ji}, b_{ji}, c_j, d_j, $d_{ji'}$, $c_{ji'}$, f_i, e_i, $f_{ji''}$, $e_{ji''}$ are complex numbers. The contour L_1 is in the s-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma^{C_j}(1 - c_j - \gamma_j s)(j = 1,.., n_2)$, $\Gamma^{A_i}(1 - a_j - \alpha_j s - A_j t)(j = 1,...., n_1)$ and $\Gamma^{E_j}(1 - e_j - E_j t)(j = 1,...., n_3)$ are to the right of L_1. The contour L_2 is in the t-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma^{D_j}(d_j + \delta_j s)$, $(j = 1,..., m_2)$, $\Gamma^{B_i}(b_j + \beta_j s + B_j t)$, $(j = 1,..., m_1)$ and $\Gamma^{F_j}(f_j + F_j t)$, $(j = 1,..., m_3)$ lie to the left of L_2. The poles of the integrand are assumed to be simple.

The function defined by the double integrals (1.1) is analytic of z_1 and z_2 if

$$U_1 = \sum_{j=1}^{P_i} \alpha_{ji}\mathbf{A}_{ji} - \sum_{j=1}^{Q_i} \beta_{ji}\mathbf{B}_{ji} + \sum_{j=1}^{P_{i'}} \gamma_{ji'}\mathbf{C}_{ji'} - \sum_{j=1}^{Q_{i'}} \delta_{ji'}\mathbf{D}_{ji'} < 0 \tag{1.5}$$

$$U_2 = \sum_{j=1}^{P_i} A_{ji}\mathbf{A}_{ji} - \sum_{j=1}^{Q_i} B_{ji}\mathbf{B}_{ji} + \sum_{j=1}^{P_{i''}} E_{ji''}\mathbf{E}_{ji''} - \sum_{j=1}^{Q_{i''}} F_{ji''}\mathbf{F}_{ji''} < 0 \tag{1.6}$$

The double integrals defined by (1.1) converges absolutely if

$$U_3 = \sum_{j=1}^{n_1} \alpha_j\mathbf{A}_i + \sum_{j=1}^{m_1} \beta_j\mathbf{B}_i + \sum_{j=1}^{n_2} \gamma_j\mathbf{C}_j + \sum_{j=1}^{m_2} \delta_j\mathbf{C}_j - \max_{1\le i\le r}\left(\sum_{j=n_1+1}^{P_i} \alpha_{ji}\mathbf{A}_{ji} + \sum_{j=m_1+1}^{Q_i} \beta_{ji}\mathbf{B}_{ji}\right)$$
$$- \max_{1\le i'\le r'}\left(\sum_{j=n_2+1}^{P_{i'}} \gamma_{ji'}\mathbf{C}_{ji'} + \sum_{j=m_2+1}^{Q_{i'}} \delta_{ji'}\mathbf{D}_{ji'}\right) > 0 \tag{1.7}$$

$$U_4 = \sum_{j=1}^{n_1} A_j\mathbf{A}_i + \sum_{j=1}^{m_1} B_j\mathbf{B}_i + \sum_{i=1}^{m_3} F_j\mathbf{F}_j + \sum_{j=1}^{n_3} E_j\mathbf{E}_j - \max_{1\le i\le r}\left(\sum_{j=n_1+1}^{P_i} A_{ji}\mathbf{A}_{ji} + \sum_{j=m_1+1}^{Q_i} B_{ji}\mathbf{B}_{ji}\right)$$
$$- \max_{1\le i''\le r''}\left(\sum_{j=n_3+1}^{P_{i''}} E_{ji''}\mathbf{E}_{ji''} + \sum_{j=m_3+1}^{Q_{i''}} F_{ji''}\mathbf{F}_{ji''}\right) > 0 \tag{1.8}$$

and we have the two inequalities: $|\arg z_1| < \frac{\pi}{2}U_3$, $|\arg z_2| < \frac{\pi}{2}U_4$.

We may establish the asymptotic behavior in the following convenient form.

$$\psi(z_1, z_2) = 0(\,|z_1|^{\alpha_1},\, |z_r|^{\alpha_2}\,),\ \max(\,|z_1|, |z_2|\,) \to 0$$

$$\psi(z_1, z_2) = 0(\,|z_1|^{\beta_1}, |z_2|^{\beta_2}\,),\ \min(\,|z_1|, |z_2\,|\,) \to \infty\ :$$

where

$$\alpha_1 = \min_{1 \leqslant j \leqslant m_2} Re\left[\left(\mathbf{D}_j \frac{d_j}{\delta_j}\right)\right] \text{ and } \alpha_2 = \min_{1 \leqslant j \leqslant m_3} Re\left[\left(\mathbf{F}_j \frac{f_j}{F_j}\right)\right]$$

$$\beta_1 = \max_{1 \leqslant j \leqslant n_2} Re\left[\left(\mathbf{C}_j \frac{1 - c_j}{\gamma_j}\right)\right] \text{and } \beta_2 = \max_{1 \leqslant j \leqslant n_3} Re\left[\left(\mathbf{E}_j \frac{1 - e_j}{E_j}\right)\right],$$

In the following, we will the notations :

$$A = (a_j, \alpha_j, A_j; \mathrm{A}_j)_{1, n_1}, [(a_{ji}, \alpha_{ji}, A_{ji}; \mathrm{A}_{ji})_{n_1+1, P_i}] : A' = (c_j, \gamma_j, \mathrm{C}_j)_{1, n_2}; [(c_{ji'}, \gamma_{ji'}; \mathrm{C}_{ji'})_{1, n_2+1, P_{i'}}], (e_j, E_j; \mathrm{E}_j)_{1, n_3}, [(e_{ji''}, E_{ji''}; \mathrm{E}_{ji''})_{n_3+1, P_{i'}}] \tag{1.9}$$

$$B = (b_j, \beta_j, B_j; \mathrm{B}_j)_{1, m_1}, [(b_{ji}, \beta_{ji}, B_{ji}; \mathrm{B}_{ji})_{m_1+1, Q_i}] : B' = (d_j, \delta_j, \mathrm{D}_j)_{1, m_2}; [(d_{ji'}, \delta_{ji'}; \mathrm{D}_{ji'})_{1, m_2+1, Q_{i'}}], (f_j, F_j; \mathrm{F}_j)_{1, m_3}, [(f_{ji''}, F_{ji''}; \mathrm{F}_{ji''})_{m_3+1, Q_{i''}}] \tag{1.10}$$

2. Main result :

We have the finite integral depending of six parameters studied by MacRobert [10],

Lemma :

$$\int_0^1 x^{\lambda}(1-x)^{\gamma-1}[1+cx+d(1-x)]^{-\lambda-\gamma} F\left[\alpha, \beta; \gamma : \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right] dx =$$

$$\frac{(1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\lambda)\Gamma(\gamma)\Gamma(\lambda+\gamma-\alpha-\beta)}{\Gamma(\lambda+\gamma-\alpha)\Gamma(\lambda+\gamma-\beta)} \tag{2.1}$$

F being the Gauss Hypergeometric function [3]. The validity conditions are the following :

Re(λ), Re(γ), Re($\gamma - \alpha - \beta$) > 0 and $1 + cx + d(1 - x) > 0$ for all x between 0 and 1.

3. Main integral :

We have the below integral involving the generalized Psi function of two variables :

Theorem :

$$\int_0^1 x^\lambda (1-x)^{\gamma-1} [1+cx+d(1-x)]^{-\lambda-\gamma} F\left[\alpha, \beta; \gamma : \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$\psi\left[z_1 \left[\frac{x}{1+cx+d(1-x)}\right]^k, z_2 \left[\frac{x}{1+cx+d(1-x)}\right]^l\right] dx = (1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\gamma)$$

$$\psi^{m_1, n_1+2; m_2, n_2 : m_3, n_3}_{P_i+2, Q_i+2; r; P_{i'}, Q_{i'}; r'; P_{i''}, Q_{i''}; r''} \left(\begin{array}{c|c} \frac{z_1}{(1+c)^k} \cdot & \mathrm{A}_2, A : A' \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_2}{(1+c)^l} & \mathrm{B}, \mathrm{B}_2 : B' \end{array} \right) \tag{3.1}$$

where

$$\left. \begin{array}{l} A_2 = (1 - \lambda; k, l; 1), (1 + \alpha + \beta - \lambda - \gamma; k, l; 1); \\ B_2 = (1 + \alpha - \lambda - \gamma; k, l; 1), (1 + \beta - \lambda - \gamma; k, l; 1) \end{array} \right\} \tag{3.2}$$

$$\text{Let } X = \frac{x}{1 + cx + d(1 - x)}$$

The conditions noted (E) are the following : $k, l > 0$ where $|\arg z_1 X^k| < \frac{1}{2} U_3 \pi$ and $|\arg z_2 X^l| < \frac{1}{2} U_4 \pi$, the quantities, U_3 and U_4 are defined respectively by (1.7) and (1.8) and $\mathrm{Re}(\lambda)$, $\mathrm{Re}(\gamma)$, $\mathrm{Re}(\gamma - \alpha - \beta) > 0$,

$$Re\,(\lambda) \;\; +k \min_{1 \leqslant j \leqslant m_2} Re\left[\left(\mathbf{D}_j \frac{d_j}{\delta_j}\right)\right] +l \min_{1 \leqslant j \leqslant m_3} Re\left[\left(\mathbf{F}_j \frac{f_j}{F_j}\right)\right] > 0$$

The numbers c and d are constants and $1 + cx + 1 - d(1 - x) > 0$ where $0 < x < 1$.

Proof : To prove the theorem, expressing the generalized Psi-function of two variables in Mellin-Barnes contour integrals with the help of (1.1). Interchanging the (s, t)-integrals and x-integral which is justifiable due to absolute convergence of the integrals involved in the process, we obtain I (left hand side of (3.1)) :

$$I = \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(s,t)\theta_1(s)\theta_2(t) z_1^{-s} z_2^{-t} \int_0^1 x^{\lambda}(1-x)^{\gamma-1}[1+cx+d(1-x)]^{-\lambda-\gamma}$$

$$\left[\frac{x}{1+cx+d(1-x)}\right]^{-ks} \left[\frac{x}{1+cx+d(1-x)}\right]^{-lt} F\left[\alpha,\beta;\gamma : \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right] \mathrm{d}x\mathrm{d}s\mathrm{d}t \tag{3.3}$$

After algebraic manipulations, we have :

$$I = \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(s,t)\theta_1(s)\theta_2(t) z_1^{-s} z_2^{-t} \int_0^1 x^{\lambda-ks-lt}(1-x)^{\gamma-1}[1+cx+d(1-x)]^{-\lambda-\gamma+ks+lt}$$

$$F\left[\alpha,\beta;\gamma : \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right] \mathrm{d}x\mathrm{d}s\mathrm{d}t \tag{3.4}$$

Now, using the lemma, we obtain :

$$I = \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(s,t)\theta_1(s)\theta_2(t) z_1^{-s} z_2^{-t} \frac{(1+c)^{-\lambda+ks+lt}(1+d)^{-\gamma}}{\Gamma(\lambda+\gamma-\alpha-ks-lt)} \frac{\Gamma(\gamma)\Gamma(\lambda-ks-lt)}{\Gamma(\lambda+\gamma-\beta-ks-lt)}$$

$$\Gamma(\lambda+\gamma-\alpha-\beta-ks-lt)\mathrm{d}s\mathrm{d}t \tag{3.5}$$

Interpreting (3.4) to the generalized Psi-function of two variables, we obtain the theorem.

4. Particular cases :

Here, we suppose that the exponents are equal to 1, we get the generalized I-function of two variables defined by M.K. Agrawal [1] and we have :

Corollary 1 :

$$\int_0^1 x^{\lambda}(1-x)^{\gamma-1}[1+cx+d(1-x)]^{-\lambda-\gamma} F\left[\alpha,\beta;\gamma : \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$I\left[z_1\left[\frac{x}{1+cx+d(1-x)}\right]^{k}, z_2\left[\frac{x}{1+cx+d(1-x)}\right]^{l}\right] dx = (1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\gamma)$$

$$I^{m_1,n_1+2;m_2,n_2:m_3,n_3}_{P_i+2,Q_i+2;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''}\left(\begin{array}{c|c} \frac{z_1}{(1+c)^k}\cdot & \mathrm{A}_2', A_1 : A_1' \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_2}{(1+c)^l} & \mathrm{B}_1, B_2' : B_1' \end{array}\right) \tag{4.1}$$

The numbers A_1, A_1', B_1 and B_1' replace respectively A, A', B and B' with the exponents are equal to 1 and we have :

$$\left.\begin{array}{l} A_2' = (1 - \lambda; k, l), (1 + \alpha + \beta - \lambda - \gamma; k, l); \\ B_2' = (1 + \alpha - \lambda - \gamma; k, l), (1 + \beta - \lambda - \gamma; k, l) \end{array}\right\} \tag{4.2}$$

The conditions have the same that the theorem with the exponents are equal to 1.

Let $r = r' = r'' = 1$, we obtain the generalized I-function of two variables defined by H. Singh and P. Kumar [14], this give the below result :

Corollary 2 :

$$\int_0^1 x^{\lambda}(1-x)^{\gamma-1}[1+cx+d(1-x)]^{-\lambda-\gamma} F\left[\alpha, \beta; \gamma : \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$I\left[z_1\left[\frac{x}{1+cx+d(1-x)}\right]^k, z_2\left[\frac{x}{1+cx+d(1-x)}\right]^l\right] dx = (1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\gamma)$$

$$I^{m_1,n_1+2;m_2,n_2:m_3,n_3}_{p_1+2,q_1+2;p_2,q_2;p_3,q_3}\left(\begin{array}{c|c} \frac{z_1}{(1+c)^k}\cdot & \mathrm{A}_2, A_{11} : A_{11}' \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_2}{(1+c)^l} & \mathrm{B}_{11}, B_2 : B_{11}' \end{array}\right) \tag{4.3}$$

The quantities A, A', B and B' are replaced respectively by A_{11}, A_{11}', B_{11} and B_{11}' where $r = r' = r'' = 1$. A_2 and B_2 are given by the equation (3.2). The conditions are the conditions (E) and $r = r' = r'' = 1$.

Supposing $r = r' = r'' = 1$ and the exponents are equal to 1, the generalized Psi function of two variables is replaced by the generalized H-function of two variables studied by Prasad and Prasad [11], this gives :

Corollary 3 :

$$\int_0^1 x^\lambda (1-x)^{\gamma-1}[1+cx+d(1-x)]^{-\lambda-\gamma} F\left[\alpha,\beta;\gamma:\frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$H\left[z_1\left[\frac{x}{1+cx+d(1-x)}\right]^k, z_2\left[\frac{x}{1+cx+d(1-x)}\right]^l\right] dx = (1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\gamma)$$

$$H^{m_1,n_1+2;m_2,n_2:m_3,n_3}_{p_1+2,q_1+2;p_2,q_2;p_3,q_3}\left(\begin{array}{c|c} \frac{z_1}{(1+c)^k}\cdot & \mathrm{A}'_2, A_{111} : A'_{111} \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_2}{(1+c)^l} & \mathrm{B}_{111}, B'_2 : B'_{111} \end{array}\right) \tag{4.4}$$

The numbers $A_{111}, A'_{111}, B_{111}$ and B'_{111} replace in this order A, A', B and B' where $r = r' = r'' = 1$ and the exponents are equals to 1. The quantities A'_2 and B'_2 are given by the equation (4.2). The conditions are verified by the corollaries 1 and 2 simultaneously.

In the below corollary 4, we suppose, $m_1 = 0$, we get the Psi function of two variables, we use this notation $B_0 = [(b_{ji}, \beta_{ji}, B_{ji}; \mathrm{B}_{ji})_{1,Q_i}]$ and we can write :

Corollary 4 :

$$\int_0^1 x^\lambda (1-x)^{\gamma-1}[1+cx+d(1-x)^{-\lambda-\gamma}] F\left[\alpha,\beta;\gamma:\frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$\psi\left[z_1\left[\frac{x}{1+cx+d(1-x)}\right]^k, z_2\left[\frac{x}{1+cx+d(1-x)}\right]^l\right] dx = (1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\gamma)$$

$$\psi^{0,n_1+2;m_2,n_2:m_3,n_3}_{P_i+2,Q_i+2;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''}\left(\begin{array}{c|c} \frac{z_1}{(1+c)^k}\cdot & \mathrm{A}_2, A : A' \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_2}{(1+c)^l} & \mathrm{B}_{01}, B_2 : B' \end{array}\right) \tag{4.5}$$

under the conditions (E) and $m_1 = 0$. A_2, B_2, A, A' and B' are defined by the theorem.

In the following, we consider $m_1 = 0$. If the exponents are equals to 1, the Psi-function of two variables is replaced by the I-function of two variables defined by Sharma and Mishra [16], we get the relation :

Corollary 5 :

$$\int_0^1 x^\lambda (1-x)^{\gamma-1}[1+cx+d(1-x)^{-\lambda-\gamma}]F\left[\alpha,\beta;\gamma:\frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$I\left[z_1\left[\frac{x}{1+cx+d(1-x)}\right]^k, z_2\left[\frac{x}{1+cx+d(1-x)}\right]^l\right]dx = (1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\gamma)$$

$$I^{0,n_1+2;m_2,n_2:m_3,n_3}_{P_i+2,Q_i+2;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r''}\left(\begin{array}{c|c} \frac{z_1}{(1+c)^k}\cdot & \mathrm{A}'_2, A_1 : A'_1 \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_2}{(1+c)^l} & \mathrm{B}'_{01}, B'_2 : B'_1 \end{array}\right) \tag{4.6}$$

Here, the exponents are equal to 1, the conditions and the notations are given by the corollary 1 and $m_1 = 0$. The quantities A'_2 and B'_2 are given by the equation (4.2). We have noted $B'_{01} = [(b_{ji}, \beta_{ji}, B_{ji})_{1, Q_i}]$.

In this situation, we have $r = r' = r'' = 1$, we have the I-function of two variables [9] and we get the following result :

Corollary 6 :

$$\int_0^1 x^\lambda (1-x)^{\gamma-1}[1+cx+d(1-x)^{-\lambda-\gamma}]F\left[\alpha,\beta;\gamma:\frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$I\left[z_1\left[\frac{x}{1+cx+d(1-x)}\right]^k, z_2\left[\frac{x}{1+cx+d(1-x)}\right]^l\right]dx = (1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\gamma)$$

$$I^{0,n_1+2;m_2,n_2:m_3,n_3}_{p_1+2,q_1+2;p_2,q_2;p_3,q_3}\left(\begin{array}{c|c} \frac{z_1}{(1+c)^k}\cdot & \mathrm{A}_2, A_{11} : A'_{11} \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_2}{(1+c)^l} & \mathrm{B}''_{01}, B_2 : B'_{11} \end{array}\right) \tag{4.7}$$

The conditions and the notations are precisely those of the corollary 2 and $m_1 = 0$. The quantities A_2' and B_2' are given by the equation (3.2) with $B_{01}'' = [(b_j, \beta_j, B_j; \mathrm{B}_j)_{1, q_i}]$.

Now, we suppose $r = r' = r'' = 1$ and the exponents are equals to 1. In this case, the Psi-function of two variables is replaced by the H-functions of two variables [5]. We have the formula :

Corollary 7 :

$$\int_0^1 x^\lambda (1-x)^{\gamma-1} [1+cx+d(1-x)^{-\lambda-\gamma}] F\left[\alpha, \beta; \gamma : \frac{(1-x)(1+d)}{1+cx+d(1-x)}\right]$$

$$H\left[z_1\left[\frac{x}{1+cx+d(1-x)}\right]^k, z_2\left[\frac{x}{1+cx+d(1-x)}\right]^l\right] dx = (1+c)^{-\lambda}(1+d)^{-\gamma}\Gamma(\gamma)$$

$$H^{0,n_1+2;m_2,n_2:m_3,n_3}_{p_1+2,q_1+2:p_2,q_2;p_3,q_3}\left(\begin{array}{c|c} \frac{z_1}{(1+c)^k}. & \mathrm{A}_2', A_{111} : A_{111}' \\ \cdot & \cdot \\ \cdot & \cdot \\ \frac{z_2}{(1+c)^l} & \mathrm{B}_{01}''', B_2' : B_{111}' \end{array}\right) \tag{4.8}$$

The conditions and notations are written in the corollary 4. The quantities A_2' and B_2' are given in the corollary 1 and we have noted : $B_{01}''' = [(b_j, \beta_j, B_j; \mathrm{B}_j)_{1, q_i}]$.

Remarks :

We have the same relations about the aleph function of one variable defined by Südland et al. [17], the aleph function of two variables studied by Sharma [15], Kumar [7], the incomplete I-function [2], the incomplete aleph function [3] and the others special functions.

Conclusion :

In this document, we have an unified single finite integral of six parameters. The importance of our all the results lies in their manifold generality. By specializing the parameters as well as variables in the generalized Psi-function of two variables, we get a big number of formulas involving remarkably wide variety of useful functions (or product of such functions) which are expressible in terms of E, F, G, H, I functions and simpler special functions of one and two variables. Secondly, concerning this single finite integral, we can have a large number of single finite integral about the special functions of two or one variables. Hence the formulae derived in this paper are most general in character and may prove to be useful in several interesting cases appearing in literature of Pure and Applied Mathematics and Mathematical Physics.

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