Applied Science Periodical
Volume - XXVII, No. 2, May 2025
Journal website: www.internationaljournalsiwan.com
ORCID Link: https://orcid.org/0009-0008-5249-8441
International Impact Factor: **7.0** https://impactfactorservice.com/home/journal/2296
Google Scholar: https://scholar.google.com/citations?user=BRweiDcAAAAJ&hl=en
Refereed and Peer-Reviewed Quarterly Periodical

ISSN 0972-5504

Multiple Finite Integrals Involving the Modified Generalized Aleph-Function of Two Variables

by **Frédéric Ayant[1],**
Teacher in High School, France

Prvindra Kumar[2], *Department of Mathematics, D.J. College, Baraut - 250611, India*

[2]Corresponding Author : prvindradjc@gmail.com

(Received: May 5, 2025; Accepted: May 23, 2025; Published Online: May 31, 2025)

Abstract:

In the present paper, we evaluate the general multiple finite integrals involving the modified generalized Aleph-function of two variables. At the end, we shall see several corollaries and remarks.

Keywords: Generalized modified Aleph-function of two variables, generalized modified I-function of two variables, generalized modified H-function of two variables, generalized modified Meijer-function of two variables, Aleph-function of two variables, I-function of two variables, H-function of two variables, Meijer G-function of two variables, two Mellin-Barnes integrals contour, multiple finite integrals

2010 Mathematics Subject Classification: 33C05, 33C60.

1. Introduction:

Recently Aleph-function of two variables has been introduced and studied by Sharma [14], Kumar [7], it's an extension of I-function of two variables defined Sharma and Mishra [15] which is a generalization of the H-function of two variables due to Gupta and Mittal [6]. On the other hand Prasad and Prasad [10] have defined the modified generalized H-function of two variables. In this paper, first, we introduce and study the generalized modified Aleph-function of two variables. This function unify the Aleph-function of two variables and the modified of generalized H-function of two variables. Later, we calculate a finite generalized integral involving this function. At the end, we will given several special cases and remarks.

The double Mellin-Barnes integrals contour occurring in this paper will be referred to as the modified generalized function of two variables throughout our present study and will be defined and represented as follows :

$$\aleph_1(z_1,z_2)=\aleph^{m_1,n_1;m_2,n_2:m_3,n_3;m_4,n_4}_{P_i,Q_i,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r'';P_{i'''},Q_{i'''},\tau_{i'''};r'''}\left(\begin{array}{c|l} z_1 & (a_j;\alpha_j,A_j)_{1,n_1}, \\ \cdot & \vdots \\ \cdot & \\ z_2 & (b_j;\beta_j,B_j)_{1,m_1}, \end{array}\right.$$

$$\begin{array}{l}[\tau_i(a_{ji};\alpha_{ji},A_{ji})]_{n_1+1,P_i};(a'_j;\alpha'_j,A'_j)_{1,n_2},[\tau_{i'}(a'_{ji'};\alpha'_{ji'},A'_{ji'})]_{n_2+1,P_{i'}}: \\ \vdots \\ [\tau_i(b_{ji};\beta_{ji},B_{ji})]_{m_1+1,Q_i};(b'_j;\beta'_j,B'_j)_{1,m_2},[\tau_{i'}(b'_{ji'};\beta'_{ji'},B'_{ji'})]_{m_2+1,Q_{i'}}:\end{array}$$

$$\left.\begin{array}{l}(c_j,\gamma_j)_{1,n_3},[\tau_{i''}(c_{ji''},\gamma_{ji''})]_{n_3+1,P_{i''}};(e_j,E_j)_{1,n_4},[\tau_{i'''}(e_{ji'''},E_{ji'''})]_{n_4+1,P_{i'''}} \\ \vdots \\ (d_j,\delta_j)_{1,m_3},[\tau_{i''}(d_{ji''},\delta_{ji''})]_{m_3+1,Q_{i''}};(f_j,F_j)_{1,m_4},[\tau_{i'''}(f_{ji'''},F_{ji'''})]_{m_4+1,Q_{i'''}}\end{array}\right)$$

$$=\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,t)\theta_1(s)\theta_2(t)\,x^s y^t\,ds\,dt \tag{1.1}$$

where

$$\phi(s,\,t)=\frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j s+A_j t)\prod_{j=1}^{m_1}\Gamma(b_j-\beta_j s-B_j t)}{\sum_{i=1}^{r}\tau_i\left[\prod_{j=m_1+1}^{Q_i}\Gamma(1-b_{ji}+\beta_{ji}s+B_{ji}t)\prod_{n_1+1}^{P_i}\Gamma(a_{ji}-\alpha_{ji}s-A_{ji}t)\right]}$$

$$\frac{\prod_{j=1}^{n_2} \Gamma(1 - a'_j + \alpha'_j s + A'_j t) \prod_{j=1}^{m_2} \Gamma(b'_j - \beta'_j s - B'_j t)}{\sum_{i'=1}^{r'} \tau_{i'} \left[\prod_{j=m_2+1}^{Q_{i'}} \Gamma(1 - b'_{ji'} + \beta'_{ji'} s + B'_{ji'} t) \prod_{n_2+1}^{P_{i'}} \Gamma(a'_{ji'} - \alpha'_{ji'} s - A'_{ji'} t) \right]} \tag{1.2}$$

$$\theta_1(s) = \frac{\prod_{j=1}^{m_3} \Gamma(d_j - \delta_j s) \prod_{j=1}^{n_3} \Gamma(1 - c_j + \gamma_j s)}{\sum_{i''=1}^{r''} \tau_{i''} \left[\prod_{j=m_3+1}^{Q_{i''}} \Gamma(1 - d_{ji''} + \delta_{ji''} s) \prod_{j=n_3+1}^{P_{i''}} \Gamma(c_{ji''} - \gamma_{ji''} s) \right]} \tag{1.3}$$

$$\theta_2(t) = \frac{\prod_{j=1}^{m_4} \Gamma(f_j - F_j t) \prod_{j=1}^{n_4} \Gamma(1 - e_j + E_j t)}{\sum_{i'''=1}^{r'''} \tau_{i'''} \left[\prod_{j=m_4+1}^{Q_{i'''}} \Gamma(1 - f_{ji'''} + F_{ji'''} t) \prod_{j=n_4+1}^{P_{i'''}} \Gamma(e_{ji'''} - E_{ji'''} t) \right]} \tag{1.4}$$

where z_1 and z_2 (real or complex) are not equal to zero and an empty product is interpreted as unity and the quantities $P_i, P_{i'}, P_{i''}, P_{i'''}, Q_i, Q_{i'}, Q_{i''}, Q_{i'''}, m_1, m_2, m_3, m_4, n_1, n_2, n_3, n_4$ are non-negative integers such that

$Q_i > 0, Q_{i'} > 0, Q_{i''} > 0, Q_{i'''} > 0; \tau_i, \tau_{i'}, \tau_{i''}, \tau_{i'''} > 0$ $(i = 1, ..., r), (i' = 1, ..., r'), (i'' = 1, ..., r''), (i''' = 1, ..., r''')$.

All the $A's$, $\alpha's$, $B's$, $\beta's$, $A''s$, $B''s$, $\alpha''s$, $\beta''s$, $\gamma's$, $\delta's$, $E's$ and $F's$ are assumed to be positive quantities for standardization purpose; the definition of generalized Aleph-function of two variables given above will however, have a meaning even if some of these quantities are zero and the numbers $a_j, b_j, a'_j, b'_j, b'_{ji'}, a'_{ji'}, b_{ji'}, a_{ji'}, c_j, d_j, d_{ji''}, c_{ji''}, f_i, g_i, f_{ji'}, g_{ji''}$ are complex numbers. The contour L_1 is in the s-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that the poles of $\Gamma(b_j - \beta_j s - B_j t)(j = 1,..., m_1)$, $\Gamma(b'_j - \beta'_j s - B'_j t)(j = 1,..., m_2)$, and $\Gamma(d_j - \delta_j s)$ $(j = 1,..., m_3)$ are to the right of L_1 and all the poles of $\Gamma(1 - a_j + \alpha_j s + A_j t)$ $(j = 1,..., n_1)$, $\Gamma(1 - c_j + \gamma s)$, $(j = 1,..., n_3)$ and $\Gamma(1 - a'_j + \alpha'_j s + A'_j t)(j = 1,..., n_2)$ lie to the left of L_1. The contour L_2 is in the t-plane and runs from $-\omega\infty$ to $+\omega\infty$ with loops, if necessary, to ensure that following poles of right of $\Gamma(b_j - \beta_j s - B_j t)$ $(j = 1,..., m_1)$, $\Gamma(b'_j - \beta'_j s - B'_j t)(j = 1,..., m_2)$ and $\Gamma(f_j - F_j t)$ $(j = 1,..., m_4)$ are to the right of L_2 and the poles of $\Gamma(1 - a_j + \alpha_j s + A_j t)(j = 1,..., n_1)$, $\Gamma(1 - a'_j + \alpha'_j s + A'_j t)$ $(j = 1,..., n_2)$ and $\Gamma(1 - e_j + E_j t)(j = 1,..., n_4)$ are to the left of L_2 lie to the left of L_2. The poles of the integrand are assumed to be simple.

The function defined by the equation (3.1) is analytic function of z_1 if and z_2 if

$$U_1 = \sum_{j=1}^{P_i} \alpha_{ji} - \sum_{j=1}^{Q_i} \beta_{ji} + \sum_{j=1}^{P_{i'}} \alpha'_{ji'} - \sum_{j=1}^{Q_{i'}} \beta'_{ji'} + \sum_{j=1}^{P_{i''}} \gamma_{ji''} - \sum_{j=1}^{Q_{i''}} \delta_{ji''} < 0 \tag{1.5}$$

$$U_2 = \sum_{j=1}^{P_i} A_{ji} - \sum_{j=1}^{Q_i} B_{ji} + \sum_{j=1}^{P_{i'}} A'_{ji'} - \sum_{j=1}^{Q_{i'}} B'_{ji'} + \sum_{j=1}^{P_{i'''}} E_{ji'''} - \sum_{j=1}^{Q_{i'''}} F_{ji'''} < 0 \tag{1.6}$$

The integral defined by (1.1) is converges absolutely, if

$$U_3 = \sum_{j=1}^{n1} \alpha_j + \sum_{j=1}^{n2} \alpha'_j + \sum_{j=1}^{m_1} \beta_j + \sum_{j=1}^{m_2} \beta'_j + \sum_{j=1}^{n_3} \gamma_j + \sum_{j=1}^{m_3} \delta_j - \max_{1\le i\le r} \tau_i \left(\sum_{j=n_1+1}^{P_i} \alpha_{ji} + \sum_{j=m_1+1}^{Q_i} \beta_{ji} \right)$$

$$- \max_{1\le i'\le r'} \tau_{i'} \left(\sum_{j=n_2+1}^{P_{i'}} \alpha'_{ji'} + \sum_{j=m_2+1}^{Q_{i'}} \beta'_{ji'} \right) - \max_{1\le i''\le r''} \tau_{i''} \left(\sum_{j=m_3+1}^{Q_{i''}} \delta_{ji''} + \sum_{j=n_3+1}^{P_{i''}} \gamma_{ji''} \right) > 0 \tag{1.7}$$

and

$$U_4 = \sum_{j=1}^{n1} \alpha_j + \sum_{j=1}^{n2} \alpha'_j + \sum_{j=1}^{m_1} \beta_j + \sum_{j=1}^{m_2} \beta'_j + \sum_{j=1}^{m_4} E_j + \sum_{j=1}^{n_4} F_j - \max_{1\le i\le r} \tau_i \left(\sum_{j=n_1+1}^{P_i} \alpha_{ji} + \sum_{j=m_1+1}^{Q_i} \beta_{ji} \right)$$

$$- \max_{1\le i'\le r'} \tau_{i'} \left(\sum_{j=n_2+1}^{P_{i'}} \alpha'_{ji'} + \sum_{j=m_2+1}^{Q_{i'}} \beta'_{ji'} \right) - \max_{1\le i'''\le r'''} \tau_{i'''} \left(\sum_{j=n_4+1}^{P_{i'''}} E_{ji'''} + \sum_{j=m_4+1}^{Q_{i'''}} F_{ji'''} \right) > 0 \tag{1.8}$$

where $|\arg z_1| < \frac{\pi}{2} U_3$ and $|\arg z_2| < \frac{\pi}{2} U_4$.

We may establish the asymptotic behavior in the following convenient form, see B.L.J. Braaksma [5]:

$$\aleph(z_1, z_2) = 0(|z_1|^{\alpha_1}, |z_r|^{\alpha_2}), \max(|z_1|, |z_2|) \to 0$$

$$\aleph(z_1, z_2) = 0(|z_1|^{\beta_1}, |z_2|^{\beta_2}), \min(|z_1|, |z_2|) \to \infty :$$

where $\alpha_1 = \min\limits_{1\le j\le m3} \mathrm{Re}\left[\left(\frac{d_j}{\delta_j}\right)\right]$ and $\alpha_2 = \min\limits_{1\le j\le m4} \mathrm{Re}\left[\left(\frac{f_j}{F_j}\right)\right]$;

$\beta_1 = \max\limits_{1\le j\le n3} \mathrm{Re}\left[\left(\frac{c_j - 1}{\gamma_j}\right)\right]$ and $\beta_2 = \max\limits_{1\le j\le n4} \mathrm{Re}\left[\left(\frac{e_j - 1}{E_j}\right)\right]$

2. Required Integral:

In this section, we give an unified multiple finite integrals. In the following, we will use the expression of the generalized Gamma-function.

Lemma: See Prudnikov et al. ([11], Ch. 3.3.3, 5 page 589).

$$\int_{x_1\geq 0}\cdots\int_{\substack{x_n\geq 0\\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}}dx_1\cdots dx_n = \Gamma\left[\begin{matrix}\frac{\upsilon_1}{\alpha_1},\ldots,\frac{\upsilon_n}{\alpha_n},1-\mu\\ \vdots \\ 1-\mu+\sum_{k=1}^{n}\frac{\upsilon_k}{\alpha_k}\end{matrix}\right]\prod_{k=1}^{n}\alpha_k^{-1} \tag{2.1}$$

provided $\alpha_k, \mu, \upsilon_k (k = 1,\ldots, n) > 0, \mu < 1$.

Now, we study an unified multiple finite integrals involving the modified generalized Aleph-function of two variables.

3. Main results:

Theorem:

$$\int_{x_1\geq 0}\cdots\int_{\substack{x_n\geq 0\\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}}\aleph\left(z_1\prod_{k=1}^{n}x_k^{\varepsilon_k}, z_2\prod_{k=1}^{n}x_k^{\mu_k}\right)dx_1\cdots dx_n = \prod_{k=1}^{n}\alpha_k^{-1}\Gamma(1-u)$$

$$\aleph^{m_1,n_1+n;m_2,n_2:m_3,n_3;m_4,n_4}_{P_i+n,Q_i+1,\tau_i;r;P_{i'},Q_{i'},\tau_{i'};r';P_{i''},Q_{i''},\tau_{i''};r'';P_{i'''},Q_{i'''},\tau_{i'''};r'''}\left(\begin{matrix}z_1\\ \vdots\\ z_2\end{matrix}\;\middle|\;\begin{matrix}\mathbf{A}_n, (a_j;\alpha_j,A_j)_{1,n_1},\\ \\ (b_j;\beta_j,B_j)_{1,m_1},\end{matrix}\right.$$

$$[\tau_i(a_{ji};\alpha_{ji},A_{ji})]_{n_1+1,P_i}; (a'_j;\alpha'_j,A'_j)_{1,n_2}, [\tau_{i'}(a'_{ji'};\alpha'_{ji'},A'_{ji'})]_{n_2+1,P_{i'}}:$$

$$[\tau_i(b_{ji};\beta_{ji},B_{ji})]_{m_1+1,Q_i},\mathbf{B}_1; (b'_j;\beta'_j,B'_j)_{1,m_2}, [\tau_{i'}(b'_{ji'};\beta'_{ji'},B'_{ji'})]_{m_2+1,Q_{i'}}:$$

$$\left.\begin{matrix}(c_j,\gamma_j)_{1,n_3}, [\tau_{i''}(c_{ji''},\gamma_{ji''})]_{n_3+1,P_{i''}}; (e_j,E_j)_{1,n_4}, [\tau_{i'''}(e_{ji'''},E_{ji'''})]_{n_4+1,P_{i'''}}\\ \vdots \\ (d_j,\delta_j)_{1,m_3}, [\tau_{i''}(d_{ji''},\delta_{ji''})]_{m_3+1,Q_{i''}}; (f_j,F_j)_{1,m_4}, [\tau_{i'''}(f_{ji'''},F_{ji'''})]_{m_4+1,Q_{i'''}}\end{matrix}\right) \tag{3.1}$$

Provided that:

$$\mu, \alpha_k, \mu_k, \upsilon_k, \varepsilon_k \ (k = 1,..., n) > 0, \ \mu < 1,$$

$$\text{with } \frac{\upsilon_k}{\alpha_k \mu} + \min_{1 \le j \le m3} \varepsilon_k \operatorname{Re}\left(\frac{d_j}{\delta_j}\right) + \min_{1 \le j \le m4} \mu_k \operatorname{Re}\left(\frac{f_j}{F_j}\right) < 1$$

where $i = 1,..., r; j = 1,..., n$

respectively by the equations (1.7) and (1.8).

$$\mathbf{A}_n = \left(1 - \frac{\upsilon_1}{\alpha_1}; \frac{\varepsilon_1 + \mu_1}{\alpha_1}\right), ..., \left(1 - \frac{\upsilon_n}{\alpha_n}; \frac{\varepsilon_n + \mu_n}{\alpha_n}\right) \tag{3.2}$$

$$\mathbf{B}_1 = \left(\mu - \sum_{k=1}^{n} \frac{\upsilon_k}{\alpha_k}; \sum_{k=1}^{n} \frac{\varepsilon_k + \mu_k}{\alpha_k}\right) \tag{3.3}$$

To prove the theorem, expressing the modified generalized Aleph-function of two variables in two Mellin-Barnes contour integral with the help of (1.4) and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Collecting the power of the x_k $(k = 1,..., n)$, this gives I

$$I = \int_{x_1 \ge 0} \cdots \int_{\substack{x_n \ge 0 \\ x_1^{\alpha_1} + \cdots + x_n^{\alpha_n} \le 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k - 1}}{(1 - x_1^{\alpha_1} - \cdots - x_n^{\alpha_n})^{\mu}} \aleph\left(z_1 \prod_{k=1}^{n} x_k^{\varepsilon_k}, z_2 \prod_{k=1}^{n} x_k^{\mu_k}\right) dx_1 \cdots dx_n$$

$$= \int_{x_1 \ge 0} \cdots \int_{\substack{x_n \ge 0 \\ x_1^{\alpha_1} + \cdots + x_n^{\alpha_n} \le 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k - 1}}{(1 - x_1^{\alpha_1} - \cdots - x_n^{\alpha_n})^{\mu}} \frac{1}{(2\pi\omega)^2}$$

$$\int_{L_1} \int_{L_2} \phi(s,t) \theta_1(s) \theta_2(t) \, z_1^s z_2^t \prod_{k=1}^{n} x_k^{s\varepsilon_k + t\mu_k} \, ds \, dt \, dx_1 \cdots dx_n \tag{3.4}$$

We have the following relation:

$$I = \frac{1}{(2\pi\omega)^2} \int_{L_1} \int_{L_2} \phi(s,t) \theta_1(s) \theta_2(t) \, z_1^s z_2^t$$

$$\int_{x_1\geq 0}\cdots\int_{x_n\geq 0}\limits_{x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k+\varepsilon_k s+\mu_k t-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}}\, dx_1\cdots dx_n\, ds dt \tag{3.5}$$

Applying the lemma, this gives:

$$I=\prod_{k=1}^{n}\alpha_k^{-1}\frac{1}{(2\pi\omega)^2}\int_{L_1}\int_{L_2}\phi(s,t)\theta_1(s)\theta_2(t)\, z_1^s z_2^t$$

$$\Gamma\left[\begin{array}{c}\frac{\upsilon_1+\varepsilon_1 s+\mu_1 t}{\alpha_1},\ldots,\frac{\upsilon_n+\varepsilon_n s+\mu_n t}{\alpha_n},\ 1-\mu\\ \vdots \\ 1-\mu+\sum_{k=1}^{n}\frac{\upsilon_k+\varepsilon_k s+\mu_k t}{\alpha_k}\end{array}\right] ds dt \tag{3.6}$$

Interpreting the generalized Gamma-function Γ, then the above formula with the help of the definition of the modified generalized Aleph-function of two variables, we have the desired equation (3.1).

4. Special Cases:

Let $\tau_i,\ \tau_{i'},\ \tau_{i''},\ \tau_{i'''}\rightarrow 1$, the generalized modified Aleph-function of two variables reduces to generalized modified I-function of two variables, this gives

Corollary 1:

$$\int_{x_1\geq 0}\cdots\int_{x_n\geq 0}\limits_{x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}}\, I\left(z_1\prod_{k=1}^{n}x_k^{\varepsilon_k},\ z_2\prod_{k=1}^{n}x_k^{\mu_k}\right) dx_1\cdots dx_n=\prod_{k=1}^{n}\alpha_k^{-1}\,\Gamma(1-\mu)$$

$$I^{m_1,n_1+n;m_2,n_2:m_3,n_3;m_4,n_4}_{P_i+n,Q_i+1;r;P_{i'},Q_{i'};r';P_{i''},Q_{i''};r'';P_{i'''},Q_{i'''};r'''}\left(\begin{array}{c|c} z_1 & \mathbf{A}_n,\ (a_j;\alpha_j,A_j)_{1,n_1},\\ \vdots & \vdots \\ z_2 & (b_j;\beta_j,B_j)_{1,m_1},\end{array}\right.$$

$$\begin{array}{l}[(a_{ji};\alpha_{ji},A_{ji})]_{n_1+1,P_i};\ (a'_j;\alpha'_j,A'_j)_{1,n_2},\ [(a'_{ji'};\alpha'_{ji'},A'_{ji'})]_{n_2+1,P_{i'}}:\\ \qquad\vdots \qquad\qquad\qquad\qquad \vdots \\ [(b_{ji};\beta_{ji},B_{ji})]_{m_1+1,Q_i},\mathbf{B}_1;\ (b'_j;\beta'_j,B'_j)_{1,m_2},\ [(b'_{ji'};\beta'_{ji'},B'_{ji'})]_{m_2+1,Q_{i'}}:\end{array}$$

$$\left. \begin{matrix} (c_j,\gamma_j)_{1,n_3}, [(c_{ji''},\gamma_{ji''})]_{n_3+1,P_{i''}} ; (e_j,E_j)_{1,n_4}, [(e_{ji'''},E_{ji'''})]_{n_4+1,P_{i'''}} \\ \vdots \\ (d_j,\delta_j)_{1,m_3}, [(d_{ji''},\delta_{ji''})]_{m_3+1,Q_{i''}} ; (f_j,F_j)_{1,m_4}, [(f_{ji'''},F_{ji'''})]_{m_4+1,Q_{i'''}} \end{matrix} \right) \tag{4.1}$$

under the same notations and conditions verified by the theorem with τ_i, $\tau_{i'}$, $\tau_{i''}$, $\tau_{i'''} \to 1$.

We consider the above corollary 1 and we suppose $r = r' = r'' = r''' = 1$, we obtain the modified generalized H-function defined by Prasad and Prasad [10] and the following integral.

Corollary 2:

$$\int_{x_1 \geq 0} \cdots \int_{\substack{x_n \geq 0 \\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n} \leq 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k - 1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}} H\left(z_1 \prod_{k=1}^{n} x_k^{\varepsilon_k}, z_2 \prod_{k=1}^{n} x_k^{\mu_k}\right) dx_1 \cdots dx_n = \prod_{k=1}^{n} \alpha_k^{-1} \Gamma(1-\mu)$$

$$H^{m_1,n_1+n;m_2,n_2:m_3,n_3;m_4,n_4}_{P+n,Q+1:P',Q':P'',Q'':P''',Q'''} \left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix} \middle| \begin{matrix} \mathbf{A}_n, (a_j;\alpha_j,A_j)_{1,P_1}; (a'_j;\alpha'_j,A'_j)_{1,P_2}: \\ \vdots \\ (b_j;\beta_j,B_j)_{1,Q_1}, \mathbf{B}_1; (b'_j;\beta'_j,B'_j)_{1,Q_2}: \end{matrix} \right.$$

$$\left. \begin{matrix} (c_j,\gamma_j)_{1,P_3}; (e_j,E_j)_{1,P_4} \\ \vdots \\ (d_j,\delta_j)_{1,Q_3}; (f_j,F_j)_{1,Q_4} \end{matrix} \right) \tag{4.2}$$

with the same notations and conditions the corollary 1 with $r = r' = r'' = r''' = 1$.

Taking $(\alpha_j)_{1,P_1} = (A_j)_{1,P_1} = (\alpha'_j)_{1,P_2} = (A'_j)_{1,P_2} = (\gamma_j)_{1,P_3} = (E_j)_{1,P_4} = (\beta_j)_{1,Q_1} = (B_j)_{1,q_1} = (\beta'_j)_{1,Q_2} = 1 = (B'_j)_{1,Q_2} = (\delta_j)_{1,Q_3} = (F_j)_{1,Q_4}$, the modified generalized H-function of two variables is replaced by the generalized modified of Meijer G-function of two variables generalizing the function defined by Agarwal [1], we have the below integrals.

Corollary 3:

$$\int_{x_1\geq 0}\cdots\int_{x_n\geq 0}\limits_{x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}}\, G\left(z_1\prod_{k=1}^{n}x_k^{\varepsilon_k}, z_2\prod_{k=1}^{n}x_k^{\mu_k}\right)dx_1\cdots dx_n=\prod_{k=1}^{n}\alpha_k^{-1}\Gamma(1-\mu)$$

$$G^{m_1,n_1+n;m_2,n_2:m_3,n_3;m_4,n_4}_{P+n,Q+1:P',Q':P'',Q'':P''',Q'''}\left(\begin{array}{c} z_1\\ \cdot\\ \cdot\\ z_2\end{array}\middle|\begin{array}{c} \mathbf{A}'_n, (a_j)_{1,P_1}: (a'_j)_{1,P_2}: (c_j)_{1,P_3}; (e_j)_{1,P_4}\\ \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\ (b_j)_{1,Q_1}, \mathbf{B}'_1: (b'_j)_{1,Q_2}: (d_j)_{1,Q_3}; (f_j)_{1,Q_4}\end{array}\right) \tag{4.3}$$

under the conditions verified by the corollary 2 and the conditions mentioned at the beginning of the corollary 3 where

$$\mathbf{A}'_n=\left(1-\frac{\upsilon_1}{\alpha_1};\frac{\varepsilon_1+\mu_1}{\alpha_1}\right),\ldots,\left(1-\frac{\upsilon_n}{\alpha_n};\frac{\varepsilon_n+\mu_n}{\alpha_n}\right) \tag{4.4}$$

$$\mathbf{B}'_1=\left(\mu-\sum_{k=1}^{n}\frac{\upsilon_k}{\alpha_k};\sum_{k=1}^{n}\frac{\varepsilon_k+\mu_k}{\alpha_k}\right) \tag{4.5}$$

and $|\arg z_1|<\frac{1}{2}U_1\pi$, $|\arg z_2|<\frac{1}{2}V_1\pi$, U_1 and V_1 are defined by the following formulas :

$$U_1=[m_1+n_1+m_2+n_2+m_3+n_3-\frac{1}{2}(p_1+q_1+p_2+q_2+p_3+q_3)] \tag{4.6}$$

and

$$V_1=[m_1+n_1+m_2+n_2+m_4+n_4-\frac{1}{2}(p_1+q_1+p_2+q_2+p_4+q_4)] \tag{4.7}$$

Let $m_2=n_2=P_{i'}=Q_{i'}=0$, the modified generalized Aleph-function of two variables reduces to generalized Aleph-function of two variables, this gives:

Corollary 4:

$$\int_{x_1\geq 0}\cdots\int_{x_n\geq 0}\limits_{x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}}\,\aleph\left(z_1\prod_{k=1}^{n}x_k^{\varepsilon_k}, z_2\prod_{k=1}^{n}x_k^{\mu_k}\right)dx_1\cdots dx_n=\prod_{k=1}^{n}\alpha_k^{-1}\Gamma(1-\mu)$$

$$\aleph^{m_1,n_1+n:m_3,n_3;m_4,n_4}_{P_i+n,Q_i+1,\tau_i;r:P_{i''},Q_{i''},\tau_{i''};r'':P_{i'''},Q_{i'''},\tau_{i'''};r'''}\left(\begin{array}{c|l} z_1 & \mathbf{A}_n,\ (a_j;\alpha_j,A_j)_{1,n_1}, \\ \cdot & \quad\vdots \\ \cdot & \\ z_2 & (b_j;\beta_j,B_j)_{1,m_1},\end{array}\right.$$

$$\begin{array}{l}[\tau_i(a_{ji};\alpha_{ji},A_{ji})]_{n_1+1,P_i}:(c_j,\gamma_j)_{1,n_3},[\tau_{i''}(c_{ji''},\gamma_{ji''})]_{n_3+1,P_{i''}};\\ \qquad\vdots \qquad\qquad\qquad\qquad\qquad\vdots \\ [\tau_i(b_{ji};\beta_{ji},B_{ji})]_{m_1+1,Q_i},\mathbf{B}_1:(d_j,\delta_j)_{1,m_3},[\tau_{i''}(d_{ji''},\delta_{ji''})]_{m_3+1,Q_{i''}};\end{array}$$

$$\left.\begin{array}{l}(e_j,E_j)_{1,n_4},[\tau_{i'''}(e_{ji'''},E_{ji'''})]_{n_4+1,P_{i'''}}\\ \qquad\qquad\vdots \\ (f_j,F_j)_{1,m_4},[\tau_{i'''}(f_{ji'''},F_{ji'''})]_{m_4+1,Q_{i'''}}\end{array}\right) \tag{4.8}$$

Under the conditions verified by the theorem and $m_2 = n_2 = P_{i'} = Q_{i'} = 0$.

Let $\tau_i, \tau_{i''}, \tau_{i'''} \to 1$, the generalized Aleph-function of two variables reduces to generalized I-function of two variables, this gives:

Corollary 5:

$$\int_{x_1\geq 0}\cdots\int_{\substack{x_n\geq 0\\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}}\ I\left(z_1\prod_{k=1}^{n} x_k^{\varepsilon_k}, z_2\prod_{k=1}^{n} x_k^{\mu_k}\right)dx_1\cdots dx_n = \prod_{k=1}^{n}\alpha_k^{-1}\Gamma(1-\mu)$$

$$I^{m_1,n_1+n:m_3,n_3;m_4,n_4}_{P_i+n,Q_i+1;r:P_{i''},Q_{i''};r'':P_{i'''},Q_{i'''};r'''}\left(\begin{array}{c|l} z_1 & \mathbf{A}_n,\ (a_j;\alpha_j,A_j)_{1,n_1},\ [(a_{ji};\alpha_{ji},A_{ji})]_{n_1+1,P_i}: \\ \cdot & \qquad\vdots\qquad\qquad\qquad\vdots \\ \cdot & \\ z_2 & (b_j;\beta_j,B_j)_{1,m_1},\ [(b_{ji};\beta_{ji},B_{ji})]_{m_1+1,Q_i},\ \mathbf{B}_1:\end{array}\right.$$

$$\left.\begin{array}{l}(c_j,\gamma_j)_{1,n_3},[(c_{ji''},\gamma_{ji''})]_{n_3+1,P_{i''}};(e_j,E_j)_{1,n_4},[(e_{ji'''},E_{ji'''})]_{n_4+1,P_{i'''}}\\ \qquad\qquad\vdots\qquad\qquad\qquad\qquad\qquad\qquad\vdots \\ (d_j,\delta_j)_{1,m_3},[(d_{ji''},\delta_{ji''})]_{m_3+1,Q_{i''}};(f_j,F_j)_{1,m_4},[(f_{ji'''},F_{ji'''})]_{m_4+1,Q_{i'''}}\end{array}\right) \tag{4.9}$$

under the same notations and conditions that corollary 4 where $\tau_i, \tau_{i''}, \tau_{i'''} \to 1$.

Consider the generalized H-function of two variables, we get:

Corollary 6:

$$\int_{x_1 \geq 0} \cdots \int_{\substack{x_n \geq 0 \\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n} \leq 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k - 1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}} H\left(z_1 \prod_{k=1}^{n} x_k^{\varepsilon_k}, z_2 \prod_{k=1}^{n} x_k^{\mu_k}\right) dx_1 \cdots dx_n = \prod_{k=1}^{n} \alpha_k^{-1} \Gamma(1-\mu)$$

$$H_{P+n,Q+1:P'',Q'':P''',Q'''}^{m_1,n_1+n:m_3,n_3;m_4,n_4}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} \mathbf{A}_n, (a_j;\alpha_j, A_j)_{1,P_1} : (c_j, \gamma_j)_{1,P_3}; (e_j, E_j)_{1,P_4} \\ \vdots \qquad \vdots \qquad \vdots \\ (b_j;\beta_j, B_j)_{1,Q_1}, \mathbf{B}_1 : (d_j, \delta_j)_{1,Q_3}; (f_j, F_j)_{1,Q_4} \end{array}\right) \tag{4.10}$$

with the same notations and conditions the corollary 2 with $m_2 = n_2 = p_2 = q_2 = 0$.

Now, we have the generalized G-function and:

Corollary 7:

$$\int_{x_1 \geq 0} \cdots \int_{\substack{x_n \geq 0 \\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n} \leq 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k - 1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}} G(z_1, z_2) dx_1 \cdots dx_n = \prod_{k=1}^{n} \alpha_k^{-1} \Gamma(1-\mu)$$

$$G_{P+n,Q+1:P'',Q'':P''',Q'''}^{m_1,n_1+n:m_3,n_3;m_4,n_4}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} \mathbf{A}'_n, (a_j)_{1,P_1} : (a'_j)_{1,P_2} : (c_j)_{1,P_3}; (e_j)_{1,P_4} \\ \vdots \qquad \vdots \qquad \vdots \\ (b_j)_{1,Q_1}, \mathbf{B}'_1 : (b'_j)_{1,Q_2} : (d_j)_{1,Q_3}; (f_j)_{1,Q_4} \end{array}\right) \tag{4.11}$$

under the conditions verified by the corollary 2 and the conditions mentioned at the beginning of the corollary 3 where

$$\mathbf{A}'_n = \left(1 - \frac{\upsilon_1}{\alpha_1}; \frac{\varepsilon_1 + \mu_1}{\alpha_1}\right), \ldots, \left(1 - \frac{\upsilon_n}{\alpha_n}; \frac{\varepsilon_n + \mu_n}{\alpha_n}\right) \tag{4.12}$$

$$\mathbf{B}'_1 = \left(\mu - \sum_{k=1}^{n} \frac{\upsilon_k}{\alpha_k}; \sum_{k=1}^{n} \frac{\varepsilon_k + \mu_k}{\alpha_k}\right) \tag{4.13}$$

and $|\arg z_1| < \frac{1}{2} U_1 \pi$, $|\arg z_2| < \frac{1}{2} V_1 \pi$, U_1 and V_1 are defined by the following formulas:

$$U_1 = [m_1 + n_1 + m_3 + n_3 - \frac{1}{2}(p_1 + q_1 + p_3 + q_3)] \tag{4.14}$$

and

$$V_1 = [m_1 + n_1 + m_4 + n_4 - \frac{1}{2}(p_1 + q_1 + p_4 + q_4)] \tag{4.15}$$

Concerning the following corollaries, we take $m_1 = 0$, and we obtain the Aleph-function defined by Sharma [14] and Kumar [7]:

Corollary 8:

$$\int_{\substack{x_1 \geq 0 \\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n} \leq 1}} \cdots \int_{x_n \geq 0} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k - 1}}{(1 - x_1^{\alpha_1} - \cdots - x_n^{\alpha_n})^{\mu}} \aleph\left(z_1 \prod_{k=1}^{n} x_k^{\varepsilon_k}, z_2 \prod_{k=1}^{n} x_k^{\mu_k}\right) dx_1 \cdots dx_n = \prod_{k=1}^{n} \alpha_k^{-1} \Gamma(1-\mu)$$

$$\aleph^{0, n_1+n : m_3, n_3; m_4, n_4}_{P_i+n, Q_i+1, \tau_i; r : P_{i''}, Q_{i''}, \tau_{i''}; r'' : P_{i'''}, Q_{i'''}, \tau_{i'''}; r'''}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_2 \end{matrix} \,\middle|\, \begin{matrix} \mathbf{A}_n, (a_j; \alpha_j, A_j)_{1, n_1}, \\ \vdots \\ [\tau_i(b_{ji}; \beta_{ji}, B_{ji})_{1, Q_i}], \mathbf{B}_1 : \end{matrix}\right.$$

$$\begin{matrix} [\tau_i(a_{ji}; \alpha_{ji}, A_{ji})]_{n_1+1, P_i} : (c_j, \gamma_j)_{1, n_3}, [\tau_{i''}(c_{ji''}, \gamma_{ji''})]_{n_3+1, P_{i''}}; (e_j, E_j)_{1, n_4}, \\ \vdots \\ (d_j, \delta_j)_{1, m_3}, [\tau_{i''}(d_{ji''}, \delta_{ji''})]_{m_3+1, Q_{i''}}; (f_j, F_j)_{1, m_4}, \end{matrix}$$

$$\left.\begin{matrix} [\tau_{i'''}(e_{ji'''}, E_{ji'''})]_{n_4+1, P_{i'''}} \\ \vdots \\ [\tau_{i'''}(f_{ji'''}, F_{ji'''})]_{m_4+1, Q_{i'''}} \end{matrix}\right) \tag{4.16}$$

Under the conditions verified by the fundamental theorem where $m_1 = m_2 = n_2 = P_{i'} = Q_{i'} = 0$ and $m_1 = 1$.

Taking $\tau_i, \tau_{i''}, \tau_{i'''} \to 1$, the Aleph-function cited above becomes the I-function of two variables defined by Sharma and Mishra [15] and we have the below relation:

Corollary 9:

$$\int_{\substack{x_1 \geq 0 \\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n} \leq 1}} \cdots \int_{x_n \geq 0} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k - 1}}{(1 - x_1^{\alpha_1} - \cdots - x_n^{\alpha_n})^{\mu}} I\left(z_1 \prod_{k=1}^{n} x_k^{\varepsilon_k}, z_2 \prod_{k=1}^{n} x_k^{\mu_k}\right) dx_1 \cdots dx_n = \prod_{k=1}^{n} \alpha_k^{-1} \Gamma(1-\mu)$$

$$I^{0,n_1+n:m_3,n_3;m_4,n_4}_{P_i+n,Q_i+1;r:P_{i''},Q_{i''};r'':P_{i'''},Q_{i'''};r'''}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} \mathbf{A}_n, (a_j;\alpha_j,A_j)_{1,n_1}, [(a_{ji};\alpha_{ji},A_{ji})]_{n_1+1,P_i}: \\ \vdots \\ [(b_{ji};\beta_{ji},B_{ji})]_{1,Q_i}, \mathbf{B}_1: \end{array}\right.$$

$$\left.\begin{array}{c} (c_j,\gamma_j)_{1,n_3}, [(c_{ji''},\gamma_{ji''})]_{n_3+1,P_{i''}}; (e_j,E_j)_{1,n_4}, [(e_{ji'''},E_{ji'''})]_{n_4+1,P_{i'''}} \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ (d_j,\delta_j)_{1,m_3}, [(d_{ji''},\delta_{ji''})]_{m_3+1,Q_{i''}}; (f_j,F_j)_{1,m_4}, [(f_{ji'''},F_{ji'''})]_{m_4+1,Q_{i'''}} \end{array}\right) \tag{4.17}$$

under the same notations and conditions that corollary 8 where $\tau_i, \tau_{i''}, \tau_{i'''} \rightarrow 1$.

Now, we consider the situation where $r = r'' = r''' = 0$, the I-function of two variables reduces to H-function of two variables defined by Gupta and Mittal [6] and we obtain:

Corollary 10:

$$\int_{x_1\geq 0} \cdots \int_{\substack{x_n\geq 0 \\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}} H\left(z_1 \prod_{k=1}^{n} x_k^{\varepsilon_k}, z_2 \prod_{k=1}^{n} x_k^{\mu_k}\right) dx_1 \cdots dx_n = \prod_{k=1}^{n} \alpha_k^{-1}\,\Gamma(1-\mu)$$

$$H^{0,n_1+n:m_3,n_3;m_4,n_4}_{P+n,Q+1:P'',Q'':P''',Q'''}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} \mathbf{A}_n, (a_j;\alpha_j,A_j)_{1,P_1}: (c_j,\gamma_j)_{1,P_3}; (e_j,E_j)_{1,P_4} \\ \vdots \qquad\qquad \vdots \\ (b_j;\beta_j,B_j)_{1,Q_1}, \mathbf{B}_1: (d_j,\delta_j)_{1,Q_3}; (f_j,F_j)_{1,Q_4} \end{array}\right) \tag{4.18}$$

with the same notations and conditions the corollary 9 with $r = r'' = r''' = 0$.

Let $m_1 = 0$; $(\alpha_j)_{1,P_1} = (A_j)_{1,P_1} = (\gamma_j)_{1,P_3} = (E_j)_{1,P_4} = (\beta_j)_{1,Q_1} = (B_j)_{1,q_1} = 1 = (\delta_j)_{1,Q_3} = (F_j)_{1,Q_4}$, the H-function of two variables is replaced by the Meijer G-function of two variables defined by Agarwal [1]. Then

Corollary 11:

$$\int_{x_1\geq 0} \cdots \int_{\substack{x_n\geq 0 \\ x_1^{\alpha_1}+\cdots+x_n^{\alpha_n}\leq 1}} \frac{\prod_{k=1}^{n} x_k^{\upsilon_k-1}}{(1-x_1^{\alpha_1}-\cdots-x_n^{\alpha_n})^{\mu}} G(z_1, z_2) dx_1 \cdots dx_n = \prod_{k=1}^{n} \alpha_k^{-1}\,\Gamma(1-\mu)$$

$$G^{0,n_1+n:m_3,n_3;m_4,n_4}_{P+n,Q+1:P'',Q'':P''',Q'''}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_2 \end{array}\middle| \begin{array}{c} \mathbf{A}'_n,\ (a_j)_{1,P_1} : (c_j)_{1,P_3} ; (e_j)_{1,P_4} \\ \vdots \qquad\qquad \vdots \\ (b_j)_{1,Q_1},\ \mathbf{B}'_1 : (d_j)_{1,Q_3} ; (f_j)_{1,Q_4} \end{array}\right) \tag{4.19}$$

under the conditions verified by the corollary 7 and the conditions mentioned at the beginning of the corollary 8 and :

$$U_1 = [n_1 + m_3 + n_3 - \frac{1}{2}(p_1 + q_1 + p_3 + q_3)] \tag{4.20}$$

and

$$V_1 = [n_1 + m_4 + n_4 - \frac{1}{2}(p_1 + q_1 + p_4 + q_4)] \tag{4.21}$$

Remarks:

We have the same generalized finite multiple integrals with the modified generalized of I-function of two variables defined by Singh and Kumar [16] generalizing the I-function of two variables studied by Kumari et al. [8] and the I-function defined by Saxena [13], the I-function defined by Rathie [12], the Fox's H-function, we have the same generalized multiple finite integrals with the incomplete aleph-function defined by Bansal et al. [3], the incomplete I-function studied by Bansal and Kumar [2] and the incomplete Fox's H-function given by Bansal et al. [4], the Psi function defined by Pragathi et al. [9].

5. Conclusion:

The importance of our all the results lies in their manifold generality. First by specializing the various parameters as well as variable of the generalized modified Aleph-function of two variables, we obtain a large number of results involving remarkably wide variety of useful special functions (or product of such special functions) which are expressible in terms of the Aleph-function of two or one variables, the I-function of two variables or one variable defined by Sharma and Mishra [15] , the H-function of two or one variables, Meijer's G-function of two or one variables and hypergeometric function of two or one variables. Secondly, by specializing the parameters of this unified multiple integrals, we can get a large number of multiple, double or single integrals involving the modified generalized aleph-functions of two variables and the others functions seen in this document.

References:

[1] R.P. Agarwal : "An extension of Meijer's G-function," Proc. Nat. Inst. Sci. India Part A, 31 (1965), 536-546.

[2] M.K. Bansal and D. Kumar : On the integral operators pertaining to a family of incomplete I-functions, AIMS Mathematics 5(2) (2020), 1247-1259.

[3] M.K. Bansal, D. Kumar, K.S. Nisar and J. Singh : Certain fractional calculus and integral transform results of incomplete Aleph-functions with applications, Math. Mech; Appli. Sci (Wiley), (2020), 1-13.

[4] M.K. Bansal, D. Kumar, I. Khan, J. Singh and K.S. Nisar : Certain unified integrals associated with product of M-series and incomplete H-functions, Mathematics, (7) (2019), 1-11.

[5] B.L.J. Braaksma : Asymptotics expansions and analytic continuations for a class of Barnes-integrals, Compositio Math. 15 (1962-1964), 239-341.

[6] K.C. Gupta, and P.K. Mittal : Integrals involving a generalized function of two variables, (1972), 430-437.

[7] D. Kumar : Generalized fractional differintegral operators of the Aleph function of two variables, Journal of Chemical, Biological and Physical Sciences, Section C, 6(3) (2016), 1116-1131.

[8] K.S. Kumari, T.M. Vasudevan Nambisan and A.K. Rathie : A study of *I*-functions of two variables, Le matematiche 69(1) (2014), 285-305.

[9] Y. Pragathi Kumar and B. Satyanarayana : A study of Psi-function, Journal of Informatics and mathematical Sciences, Vol. 12(2) (2020), 159-171.

[10] Y.N. Prasad and S. Prasad, (1979 -1980) : Journal of scientific research, Banaras Hindu University, 30(1), 67 -76.

[11] A.P. Prudnikov, Y.A. Brychkow and O.I. Marichev : Elementary functions, Integrals and Series, Vol.1, U.S.S.R. Academy of Sciences, Moscou, 1986, (Fourth printing 1998).

[12] A.K. Rathie : A new generalization of generalized hypergeometric functions, Le Matematiche, 52(2) (1997), 297-310.

[13] V.P. Saxena : The I-function, Anamaya Publishers, New Delhi, 2008.

[14] K. Sharma : On the Integral Representation and Applications of the Generalized Function of Two Variables, International Journal of Mathematical Engineering and Science, 3(1) (2014), 1-13.

[15] C.K. Sharma and P.L. Mishra : On the I-function of two variables and its Certain properties, Acta Ciencia Indica,17 (1991), 1-4.

[16] H. Singh and P. Kumar : Finite integral formulas involving multivariable Mittag-Leffler function and Modified I-function, Int. Jr. of Mathematical Sciences and Application, Vol. 8(2), (2018), 115-128.