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A Brief Study of Continued Fractions in Historical Perspective

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Abstract :

In this paper, we deal with continued fraction in Historical perspective.

Keywords : Continued fractions, infinite product, recurrence relation.

1. Introduction :

Continued fraction have attracted mathematicians since several centuries and are still relevant. We discuss the history of the theory of continued fraction. It is used in indeterminate equation.

2. Continued fractions in historical perspective :

At first glance it is very easy to write continued fractions e.g. if, we consider $\frac{9}{7}$ then

$$\frac{9}{7} = 1 + \frac{2}{7} = 1 + \frac{1}{\frac{7}{2}} = 1\frac{1}{3 + \frac{1}{2}} = 1\frac{1}{3 + \frac{1}{1 + \frac{1}{1}}}$$

But it turns out that this form of writing numbers which we call continued fractions are useful in solving many problems involving numbers and provide much insight.

These were studied in 17th and 18th centuries by great mathematicians of those days and are still a subject of modern research.

Euclid's method of finding the g.c.d of two numbers is essentially that of converting a fraction into a continued fraction. This happened in 300B.C and was an important step in the development of the theory of continued fraction. It is also mentioned in the work of the Indian mathematician Aryabhata who lived up to 550 A.D. and worked in Pataliputra or modern Patna. He used continued fraction to find general solution of a linear indeterminate equation. Occasionally in Arab and Greek writings traces of the general concept of a continued fraction have been found.

Now it is generally agreed that the modern theory of continued fractions began with the writings of Rafael Bombelli (Bron C. 1530) a native of Bologna in Italy. His treatise on algebra (1572) contains a chapter on square roots. In modern rotation he showed, for example that

$$\sqrt{13} = 3 + \frac{4}{6+}\,\frac{4}{6+}$$

This means that he had knowledge of more general case i.e.

$$\sqrt{a^2+b} = a + \frac{b}{2a+}\,\frac{b}{2a+\cdots}$$

The next writer to consider these fractions was Pietro Cataldi (1548-1626), also a native of Bologna. His work included the development of continued fractions and a method for their representation.

He expressed $\sqrt{18}$ in the form

$$\sqrt{18} = 4 + \frac{2}{8} \text{ and } \frac{2}{8} \text{ and } \frac{2}{8}$$

He modified it, for convenience in printing into the form $4\ \&\frac{2}{8}\ \&\frac{2}{8}\ \&\frac{2}{8}$, which in modern form is $4 + \frac{2}{8+}\,\frac{2}{8+}\,\frac{2}{8+\cdots}$

Another early contributor was German mathematician Daniel Schweitzer (1585-1636) who was also a professor at the university of Altdorf, Germany. In his book Geometrica Practica he obtained approximations to $\frac{177}{233}$ by finding the g.c.d. of 177 and 233 and from these computations he determined he different convergent $\frac{79}{104}, \frac{19}{25}, \frac{3}{4}, \frac{1}{1}$ and $\frac{0}{1}$. Another writer to use continued fractions was the first president of the Royal Society, Lord Brouncker (1620-1684). His academic advisor was John Wallis who discovered the infinite product.

$$\frac{4}{\pi} = \frac{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot\dots\dots\dots}{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot\dots\dots\dots}$$

Brouncker transformed it into the continued fraction $\frac{4}{\pi} = 1 + \frac{1^2}{2+} \frac{3^2}{2+} \frac{5^2}{2+} \frac{7^2}{2+} \dots$

John Wallis published his book 'Arithmetica Infinitorum' in 1655. In his book he discussed Brouncker's fraction. He also stated many of the elementary properties of the convergent to general continued fraction and also the rule for their formation. The name 'continued fraction' was also first used by him.

The great Dutch mathematician and astronomer Christiaan Huyghens (1629-1695) used continued fraction to approximate the correct design for the toothed wheels of a planetarium, which is described in his treatise description Automatic planetarium published after his demise.

The foundation of the modern theory was laid by Euler (1707-1783) in his great memoir De Fractionibus continuis (1737). Other who also contributed included Lambert (1728-1777) who gave the first proof that π is irrational using a generalised continued fraction of the function $\tan x$.

He show that

$$\tan x = \frac{1}{\frac{1}{x}-} \frac{1}{\frac{3}{x}-} \frac{1}{\frac{5}{x}-} \frac{1}{\frac{7}{x}-\dots}$$

He concluded that if x is a rational number, not 0, then $\tan x$ cannot be rational. Thus since $\tan\left(\frac{\pi}{4}\right) = 1$, π cannot be rational. In 1770, he proved that

$$\pi = 3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+\ldots}$$

$$= [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2.....]$$

Some weaknesses in Lambert's proof were corrected by Legendre in his Elements de geometrie in 1794. Legendre was a French mathematician and his book was the leading elementary text on the topic for around 108 years. Lagrange (1736-1813) was an Italian mathematician who also made contributions to the theory of continued fractions. In 1776, he has proved that when $|x| < 1$

$$(1 + x)^k = \frac{1}{1-}\frac{kx}{1+}\frac{1.(1+k)x}{\frac{1.2}{1+x}}\frac{1.(1-k)x}{\frac{2.3}{1+}}\frac{2(2+k)x}{\frac{3.4}{1+}}\frac{2(2-k)x}{\frac{4.5}{1+}}\frac{3(3+k)x}{\frac{5.6}{1+\ldots}}$$

In 1813 he showed that when $|x| < 1$.

$$\log\frac{1+x}{1-x} = \frac{2x}{1-}\frac{1.x^2}{3-}\frac{4x^2}{5-}\frac{9x^2}{7-}\frac{16x^2}{9-\ldots}$$

In 1892, Henri Padé define Padé approximation techniques for function.

In 1972, Bill Gosper found first exact algorithms for continued fraction arithmetic.

Even in present day modern mathematics continued fraction has an important role. They are still used for new discoveries in the theory of numbers and / the field of Diophantine approximations. There is a generalization of continued fraction called the analytic theory of continue fractions which is an extensive area for present and future research. It has also applications in computer field where it is used to give approximations to various complicated function. Once coded for the electronic machines, it gives rapid numerical result which are valuable to scientists and to those working in applied mathematical fields.

2.1 Continued Fractions and Indeterminate Equations :

First let us recall some basics of continued fractions. An expression of the form

$$a_1 + \cfrac{b_1}{a_2 + \cfrac{b_2}{a_3 + \cfrac{b_3}{a_4 + \cdots\cdots\cdots}}}$$

is called a continued fraction. Where is general number $a_1, a_2, a_3, \ldots, b_1, b_2, b_3, \ldots,$ may be any real or complex numbers and the number to terms can be finite or infinite. But here we restrict ourselves to simple continued fraction which have the form

$$a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cfrac{1}{a_{n-1} + a_n}}}}$$

This can abbreviated in the form

$$[a_1, a_2, \ldots., a_n] = a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \cdots \frac{1}{a_n}$$

Every rational number (from $\frac{p}{q}$, where p and q are integers with $(q \neq 0)$ can be expressed as a simple finite continued fraction e.g.

$$\frac{67}{29} = 2 + \frac{1}{3 +} \frac{1}{4 +} \frac{1}{2} \; [2, 3, 4, 2]$$

and $\frac{29}{67} = [0, 2, 3, 4, 2]$ here $a_1 = 0$

$$\frac{-37}{44} = [-1, 6, 3, 2], \text{ here } a_1 \text{ is negative integer.}$$

Also any finite simple continued fraction represents a rational number.

Now consider a rational number $\frac{p}{q}$ and expand it into a finite simple continued fraction

$$\frac{p}{q} = [a_1, a_2, ..., a_{n-1}, a_n]$$

Then $a_1, a_2, ..., a_n$ are called Quotient of the continued fraction. From these we can form it the fractions.

$$c_1 \ \frac{a_1}{1}, c_2 \quad a_1 \ \frac{1}{a_2}, c_3 \quad a_1 \ \frac{1}{a_2} \frac{1}{a_3}$$

By cutting off the expansion process after the first, second, third, steps these fractions are called the first, second, third, convergent, respectively, of the continued fraction. So the continued fractions is same as the n the convergent for

$$[a_1, a_2, ..., a_n] = a_1 \ \frac{1}{a_2} \ \frac{1}{a_n} \ c_n$$

Also we write $c_1 \ \frac{p_1}{q_1} \ \frac{a_1}{1} \Rightarrow p_1 = a_1, q_1 = 1$

Write $c_2 \ \frac{p_2}{q_2} \ a_1 \ \frac{1}{a_2} \ \frac{a_1 a_2}{a_2} \Rightarrow p_2 = a_1 a_2 + 1, q_2 = a_2$

Similarly $c_3 \ \frac{p_3}{q_3} \ a_1 \ \frac{1}{a_2 \ \frac{1}{a_3}} \frac{a_1 a_2 a_3}{a_2 a_3} \frac{a_1 a_3}{1}$ etc.

Here $p_3 = a_1 a_2 a_3 + a_1 + a_2$

$$= a_3 (a_1 a_2 + 1) + a_1 = a_3 p_2 + p_1$$

and $q_3 = a_2 a_3 + 1 = a_3 q_2 + q_1$

In general, recurrence relations are

$$\frac{p_i}{q_i} \frac{a_i p_{i-1}}{a_i q_{i-1}} \frac{p_{i-2}}{q_{i-2}} \quad ; i = 3, 4, 5, ..., n$$

With the initial values $\frac{p_1}{q_i} \frac{a_i, p_2}{1, q_2} \frac{a_1 a_2}{a_2}$

Also $c_k = [a_1, a_2, ..., a_{k-1}, a_k] = \frac{p_k}{q_k} \frac{a_k p_{k-1}}{a_k q_{k-1}} \frac{p_{k-2}}{q_{k-2}}$

We also make the conversion that

$$p_0 = 1, \; p_{-1} = 0$$

$$q_0 = 0, \; q_{-1} = 1$$

We find that these satisfy our recurrence relations for $i = 1, 2$

as $\quad c_1 = \frac{p_1}{q_1} \frac{a_1 p_0}{a_1 q_0} \frac{p_1}{q_1} \frac{a_1 . 1}{a_1 . 0} \frac{0}{1} \frac{a_1}{1}$

and $\quad c_2 = \frac{p_2}{q_2} \frac{a_1 p_1}{a_2 q_1} \frac{p_0}{q_0} \frac{a_2 a_1}{a_2 . 1} \frac{1}{0} \frac{a_1 a_2}{a_2}$

Consider $\frac{120}{49}$, if we convert it into continued fraction, we get

$$\frac{120}{49} = [2, 2, 4, 2, 2] = [a_1, a_2, a_3, a_4, a_5]$$

Here $\frac{p_1}{q_1} \frac{2}{1}$ i.e. $p_1 = 2, q_1 = 1$

$$\frac{p_2}{q_2} \frac{1}{2} \frac{5}{2} \Rightarrow p_2 = 5, q_2 = 2$$

Hence $p_3 = a_3 p_2 + p_1 = 4 \times 5 + 2 = 22$

and $\quad q_3 = a_3 q_2 + q_1 = 4 \times 2 + 1 = 9$

Next $p_4 = a_4 p_3 + p_2 = 2 \times 22 + 5 = 44 + 5 = 49$

and $q_4 = a_4 q_3 + q_2 = 2 \times 9 + 2 = 18 + 2 = 20$

Finally $p_5 = 120$ and $q_5 = 49$

So from the following table

i	-1	0	1	2	3	4	5
a_i			2	2	4	2	2
p_i	0	1	2	5	22	49	120
q_i	1	0	1	2	9	20	49
$c_i\ \frac{p_i}{q_i}$			$\frac{2}{1}$	$\frac{5}{2}$	$\frac{22}{9}$	$\frac{49}{20}$	$\frac{120}{49}$

Explanation of table :

The entries in the first row of the table are the value of i which starts with -1, 0 and ends in 5. Under each value of i we write the corresponding values of a_i, p_i, q_i and c_i.Thus under $i = 3$ we have $a_3 = 4$, $p_3 = 22$, $q_3 = 9$ and $c_3 = \frac{22}{9}$

We have some interesting results like if $a_1 \neq 0$, then

$$\frac{p_n}{p_{n-1}} [a_n, a_{n-1}, a_{n-2}, \ldots.., a_1]$$

and $$\frac{q_n}{q_{n-1}} [a_n, a_{n-1}, a_{n-2}, \ldots.., a_2]$$

We have also the following result regarding differences of convergents viz.

$$p_i q_{i-1} - p_{i-1} q_i = (-1)^i, i \geq 0$$

Let us put $i = 0$, then

$$p_0 q_{-1} - p_{-1} q_0 = 1.1 - 0.0 = 1 = (-1)^0$$

Put $i = 1$, then $p_1 q_0 - p_0 q_1 = a_1 0 - 1.1 = -1 = (-1)^1$ etc.

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