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A Study of Some Non-Linear Partial Differential Equations

by **Deepak Kumar,** *Research Scholar,*
Department of Physics,
B.N. Mandal University, Madhepura - 852113, India
E-mail : dkchandi1965@gmail.com
&
Bipin Kumar Singh, *Associate Professor & H.O.D.,*
Department of Physics,
M.L.T. College, Saharsa - 852201, India
(A constituent unit of B.N.M.U, Madhepura)

Abstract :

Discrete mathematics is an essential branch of Applied Mathematics. The Fourier series, the founding principle behind the field of Fourier analysis. The solutions of non-linear partial differential equations play an important role in the study of many physical phenomena. The fields of electronics, quantum mechanics and electrodynamics make wide use of the Fourier series. In this paper, we use the concept of Fourier series to solve the non-linear partial differential equation.

Introduction :

One of the most exciting advances of non-linear science and theoretical physics has been the development of methods to look for solutions of non-linear partial differential equations. Recently a variety of powerful methods, such as, tan$h$ - sec$h$ method [1-5], hyperbolic function method, Jacobi elliptic function expansion method, F-expansion method, and the First Integral method etc. This method has been used to solve different types of nonlinear systems of Partial Differential Equations. In this paper, we applied the Fourier series method to solve the nonlinear partial differential equations.

We represent one of the methods of solving Partial differential equation by the use of Fourier series. In this method consider the homogeneous heat equation defined on a rod of length $2l$ with periodic boundary conditions. In mathematical term we must find a solution $u = u(x, t)$ to the problem

$$\left.\begin{array}{ll} u_t - ku_{xx} = 0, & -l < x < l, 0 < t < \infty \\ u(x, 0) = f(x), & -l \leq x \leq l, \\ u(-l, t) = u(l, t), & 0 \leq t < \infty \\ u_x(-l, t) = u_x(l, t), & 0 \leq t < \infty \end{array}\right\} \quad (1)$$

where $k > 0$ is a constant. The common wisdom is that this mathematical equations model the heat flow $u(x, t)$ is the temperature in a ring 21, where the initial $(t = 0)$ distribution of temperature in the ring given by the function f. A point in the ring is represented by a point in the interval $[-l, l]$ where the end points $x = l$ and $x = -l$ represents same point in the ring. For this reason the mathematical representation of the problem includes the equations $u(-l, t) = u(l, t)$ and $ux(-l, t) = ux(l, t)$. To obtained a good solution of this problem, it is better if we assume that f is continuous, $f' \, l \in E$, and f satisfies $f(-l) = f(l)$ and $f'(-l) = f(l)$. The idea behind this method is first to find all non identical zero solution of the form $u(x, t) = X(x)\, T(t)$ to the homogeneous system

$$\left.\begin{array}{ll} u_t - ku_{xx} = 0, & -l < x < l, 0 < t < \infty \\ u(-l, t) = u(l, t), & 0 \leq t < \infty \\ u_x(-l, t) = u_x(l, t), & 0 \leq t < \infty \end{array}\right\} \quad (2)$$

Taking into consideration the system and the fact that $u(x, t) = X(x)T(t)$.

Then

$$u_t(x, t) = X(x)T'(t),$$

$$u_{xx}(x, t) = X''(x)T(t)$$

Substituting these forms in the equations, we obtain

$$X(x)T'(t) - k\,X''(x)T(t) = 0$$

and thus

$$X(x)T'(t) = kX''(x)T(t)$$

Dividing both side of the equation by $kX(x)T(t)$. We obtain

$$T'(t) / kT(t) = X''(x) / X(x)$$

The expression on the left-hand side is a function of t alone; the expression on the right-hand side is a function of x.

We already know that x and t are independent upon each other, the equation that is given above can hold only if and only if both sided of it is equal to some unknown constant $-\lambda$ for all value of x and t. Thus we may write

$$T'(t) / kT(t) = X''(x) / X(x) = -\lambda$$

Clearly, we obtain one pair differential equations with unknown constant λ

$$X''(x) + \lambda X(x) = 0$$

$$T'(t) + k\lambda\, T(t) = 0$$

From those two boundary conditions, we derive two conditions. From the boundary condition $u(-l, t) = u(l, t)$ it follows that for all $t > 0$

$$X(-l)T(t) = X(l)T(t)$$

There exist two possibilities, either $T(t) = 0$ for all $t > 0$, $X(-l) = X(l)$.

We look to the second condition $X(-l) = X(l)$. Similarly, we obtain the second condition $X'(-l) = X'(l)$. When we are looking for non trivial solutions of (2) of the form $u(x, t) = X(x)T(t)$ to the equations for X:

$$\left.\begin{array}{ll} X''(x) + \lambda X(x) = 0, & 0 < x < l, \\ X(-l) = X(l) & \\ X'(-l) = X'(l) & \end{array}\right\} \quad (3)$$

We can easily check that values of λ for which equation (3) has non-trivial solution are exactly

$$\lambda^n = n^2\pi^n/l^2, \qquad n = 0, 1, 2, \ldots\ldots\ldots$$

From the condition $X(-l) = X(l)$ we obtain $c_1 = 0$, while the condition $X'(-l) = X'(l)$ is always satisfied. This being so, in this case, the constant function $X(x) = C$ are solutions of (3). For $\lambda_n = n^2\pi^2/l^2$, $n \geq l$, the equation is

$$X''(x) + (n_2\pi_2/l_2)X(x) = 0$$

General solution has the form of

$$X(x) = c_1\sin(n\pi/l)x + c_2\cos(n\pi/l)x.$$

Finally, we have two non-trivial linearly independent solution for all $n \in N$ and $\lambda_n = n^2\pi^2/l^2$

$$X_n(x) = \cos(n\pi/l)x,$$
$$X_n^*(x) = \sin(n\pi/l)x.$$

Every other solution is a linear combination of these two solutions. The values λ_n are called the eigenvalues of the problem, and the solution of X_n and X_n^* are called the eigenfunctions associated with the eigenvalue λ_n. We also recall that among the eigenvalues, we also have $\lambda_0 = 0$, with associated eigenfunction

$$X_0(x) = l$$

Now, we consider the second equation $T'(t) + k(\lambda)T(t) = 0$. We restrict to our self to $\lambda = \lambda_n = n^2\pi^2/l^2$, $n = 0, 1, 2, 3, \ldots\ldots\ldots\ldots$ for each n there exists non-trivial solution

$$T_n(t) = e^{-k\lambda nt}$$

Every other solution is a constant multiple therefore, so, finally we can summarize, for each $n \in N$, we have pair for non-trivial solution of the form

$$u_n(x, t) = X_n(x)T_n(t) = e^{-k\lambda nt} \cos(n\pi/l)x$$
$$u_n^*(x, t) = X_n^*(x)T_n(t) = e^{-k\lambda nt} \sin(n\pi/l)x$$

For $n = 0$, we have the solution

$$u_0(x, t) = X_0(x)T_0(t) = l$$

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