The Mathematics Education **ISSN 0047-6269**
Volume - LV, No. 3, September 2021
Refereed and Peer-Reviewed Quarterly Journal
Journal website : www.internationaljournalsiwan.com

Analysis on the Fourier series for the Complex Exponential

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Abstract :

Fourier series is an infinite sum of trigonometric functions that can be used to model real-valued, periodic function. The know objective in this paper, is to study on the Fourier series of the complex exponentials. The important properties of Fourier series are described and proved, and their relevance is explained.

Introduction :

Fourier series are of great importance in both theoretical and applied mathematics. For orthonormal families of complex-valued functions $\{\phi_n\}$, Fourier series are sums of the ϕ_n that can approximate periodic complex-valued functions with arbitrary precision. There are many possible methods [1-5] of estimating complex-valued functions. The complex exponentials are relatively simple to dead with and ubiquitous in physical phenomena. We shall give a description of general Fourier Functions. We shall give a description of General Fourier functions. We start with the notion of orthogonal systems of functions. Let $\{\phi_1, \phi_2, \phi_3...\}$ be a series of complex functions. We say that $\{\phi_n\}$ is an orthogonal system of functions on $[a, b]$ if, for all integers m, $m \neq n$,

$$\int_a^b \phi_m(x)\bar{\phi}_n(x)dx = 0 \tag{1}$$

As a further note, if for all integers $m > 0$,

$$\int_a^b \phi_m(x)\bar{\phi}_m(x)dx = 1$$

We say that $\{\phi_n\}$ is an orthonormal system of functions. We know already seen that the functions e^{inx}, $n = 1, 2, 3....$ form an orthogonal system of functions on $[-\pi, \pi]$, since $e^{-einx} = e^{-inx}$, and for $m \neq n$, $\int_{-\pi}^{\pi} e^{inx} - e^{-inx} = 0$. We now define the Fourier coefficients with respect $\{\phi_n(x)\}$ as follows:

$$c_n = \int_a^b f(x)\bar{\phi}_n(x) \tag{2}$$

Where $\phi_n(x)$ is the complex conjugate of the complex-valued function $\phi(x)$. In terms of generalized Fourier series, define the Fourier series of with respect to $\{\phi_n(x)\}$ to be

$$\sum_{n=1}^{\infty} c_n\phi_n(x) \tag{3}$$

To see how our definition for the Fourier series with respect to trigonometric function matches this pattern, let

$$\{\phi_n(x)\} = \{\phi_1(x), \phi_2(x), \phi_3(x), \phi_4(x)\}$$

$$= \{e^{ix}, e^{(-ix)}, e^{2ix}, e^{(-2ix)}\}$$

and let $\phi(x) = e^{inx}$, $\bar{\phi}(x) = e^{(-inx)}$.

Some Properties of Fourier Series :

Theorem 1 in Rudin : Suppose that $\{\phi_n\}$ is an orthonormal system of function on the interval $[-\pi, \pi]$. Suppose that we have two sets of complex numbers,

c_n and d_n, $n = 0, 1, 2, 3.....$ and d_n, are the Fourier Coefficients for $\{\phi_n\}$, as defined in equation (2). Now, consider two series of functions,

$$s_N(f, x) = \sum_{n=1}^{N} c_n\{\phi_n\} \tag{4}$$

Which is the N^{th} partial sum of the Fourier series for f, and

$$t_N(f, x) = \sum_{n=1}^{N} d_n\{\phi_n\} \tag{5}$$

Then,

$$\int_a^b |f - s_n(f, x)|^2\, dx \le \int_a^b |f - t_n(f, x)|^2\, dx \tag{6}$$

This theorem indicates that, for some periodic function f and some orthonormal system of function $\{\phi_n\}$, the Fourier series provides the least total squared-error approximation.

Proof : Let $\{\phi_n(x)\}$ be orthonormal on the interval $[a, b]$. Consider

$$\int_a^b f\bar{t}_n = \int_a^b f \sum_1^n \bar{d}_n\bar{\phi}_n$$

by the definition of t_n. We can also write the above as:

$$\int_a^b f \sum_1^n \bar{d}_n\bar{\phi}_n = \sum_1^n \int_a^b f\bar{d}_n\bar{\phi}_n$$

Since, $\int_a^b f\phi_n = c_n$, we can once again rewrite the above expression as:

$$\int_a^b f\bar{t}_n = \sum_1^n c_n\bar{d}_n \tag{7}$$

Now, consider the integral from a of $|t_n|^2$. Since

$$|t_n|^2 = t_n\bar{t}_n$$

and

$$t_n = \sum_1^n d_m\phi_m$$

$$\bar{t}_n = \sum_1^n \bar{d}_m\bar{\phi}_m$$

We have,

$$\int_a^b |t_n|^2 = \int_a^b \sum_1^n d_m\phi_m \sum_1^n \bar{d}_k\bar{\phi}_k$$

Which we many rewrite as:

$$\int_a^b |t_n|^2 = \sum_1^n d_m\phi_m \int_a^b \sum_1^n \bar{d}_k\bar{\phi}_k \tag{8}$$

Since $\{\phi_n\}$ is an orthonormal system of function on $[a, b]$, according to equation (1), we can again rewrite as,

$$\int_a^b |t_n|^2 = \sum_1^n d_m\bar{d}_m$$

Which we can again rewrite as,

$$\int_a^b |t_n|^2 = \sum_1^n |d_m|^2 \tag{9}$$

Now, consider the total squared error between f and t_n, $\int_a^b |f - t_n|^2$. We first rewrite it as:

$$\int_a^b |f - t_n|^2 = \int_a^b (f - t_n)(\overline{f - t_n})$$

Furthermore, we know that

$$(\overline{f - t_n}) = \bar{f} - \bar{t}_n,$$

so that

$$\begin{aligned}\int_a^b |f - t_n|^2 &= \int_a^b (f - t_n)(\bar{f} - \bar{t}_n)\\ &= \int_a^b (f\bar{f} - f\bar{t}_n - \bar{f}t_n + t_n\bar{t}_n)\\ &= \int_a^b (f^2 - f\bar{t}_n - \bar{f}t_n + t_n^2)\end{aligned} \tag{10}$$

By equations (7) and (9) above, we can write:

$$\int_a^b |f - t_n|^2 = \int_a^b (f^2) - \sum_1^n (c_m\bar{d}_m) - \sum_1^n (\bar{c}_m d_m) + \sum_1^n (d_m\bar{d}_m)$$

Since

$$|d_m - c_m|^2 = (d_m - c_m)(\bar{d}_m - \bar{c}_m)$$
$$= |d_m|^2 + |c_m|^2 - (c_m\bar{d}_m) - (\bar{c}_m d_m)$$

We can rewrite the above equation as:

$$\int_a^b |f - t_n|^2 = \int_a^b (f^2) - \Sigma_1^n |c_m|^2 + \Sigma_1^n |d_m - c_m|^2 \tag{11}$$

From this above equation, we can see that the total error squared is minimized when $d_m = c_m$, for $m = 1, 2, 3....$

Theorem 8.12 in Rudin[1] : Assume all the notation used in the description of Theorem 8.11. Consider the sequence of terms $\{c_n\} = c_1, c_2, c_3$...The series 1, Σ_1^n (c_m) converges absolutely (in other words, the series $\Sigma_1^n |c_m|$ converges).

$$\int_a^b |s_n|^2 (x)dx = \Sigma_1^n |c_m|^2 \tag{12}$$

The above step will not be necessary, but it is interesting to point out that the integral of the absolute value of any n^{th} Fourier trigonometric polynomial will be less than the integral of the absolute value of the function f. Now consider (11), since we know that $\int_a^b |f - t_n|^2 \geq 0$, it follows that

$$\Sigma_1^n |c_m|^2 \leq \int_a^b |f(x)|^2 dx \tag{13}$$

If we let n go to infinity, we see that:

$$\Sigma_1^\infty |c_n|^2 \leq \int_a^b |f(x)|^2 \tag{14}$$

From our study of convergent series, this also implies that

$$\lim_{n\to\infty} c_n = 0$$

This result is fairly important because it shows us that it is the first terms of a Fourier series that are most important, and that the Fourier coefficients become arbitrarily small. In terms of simulations, this implies that a few terms may provide a very good model of a function.

Conclusion :

With these new techniques, Fourier series and Transforms have become an integral part of the toolboxes of mathematicians and scientists. Today, it is used for applications as diverse as file compression (such as the JPEG image format), signal processing in communications and astronomy, acoustics, optics, and cryptography.

Acknowledgement :

We are grateful to Dr. Surendra Kumar Roy, Academic incharge & HOD Applied Science, Supaul College of Engineering, Supaul for his valuable suggestions and many fruitful discussions regarding to take up this project.

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