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# Spacetimes with Polycrystalline Symmetry

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**Abstract :**

*We find the conditions imposed on the metric of a spacetime by requiring that its spacelike hypersurfaces possess, at a given scale, more than one Euclidean crystalline symmetry. Such spacetimes gather into four classes. We show that three of these classes correspond to spacetimes without physical meaning. It follows that no material object can be created with the perfect crystalline structure belonging to these three classes. This result implies contradictions between general relativity and crystallography.*

## 1. Introduction :

The classical theory of general relativity is commonly believed to apply from the cosmological to the Planck scales, where it should be replaced by a still elusive theory of quantum gravity. But the domain of applicability of general

relativity could be not so well delimited. We have indeed shown in [1] that general relativity and quantum mechanics cannot be applied simultaneously at the atomic or molecular scales without intrinsic contradictions about their predictions, even if the spacetime is everywhere of weak curvature. This results from the fact that every spacetime determined by a nonspherically symmetric body with the symmetry of any one of the five Platonic solids has a curvature tensor which vanishes in a neighbourhood of the centre of this symmetry. Such a body must then have an empty centre of symmetry.

In the present paper, we investigate the consequences of assuming that the space part of a spacetime has a symmetry which often applies to bodies of a scale just above the one of molecules, *viz* a crystalline symmetry. Within classical general relativity, we shall determine the conditions imposed on the metric of a spacetime by requiring that its spacelike hypersurfaces possess, at a given scale, more than one Euclidean crystalline symmetries. Spacetimes of this kind will be said to have a polycrystalline symmetry. The $3D$ Euclidean crystallography shows that there are four classes of polycrystalline symmetries. Using purely geometric arguments, we shall obtain the general form of the metric for spacetimes of each of these classes. This will allow us to show that three of these classes correspond to spacetimes without physical meaning. It follows that no material object can be created with the exact polycrystalline symmetry of any one of the first three classes. We shall see that this contradicts results from crystallography. The general form of the curvature tensor for spacetimes of the fourth class is also determined. The general forms of the metric and the curvature tensor for spacetimes of the fourth class could be used as starting points for the determination, through Einstein's equations, of the effective metric of any spacetime with this kind of symmetry.

We have organized this paper as follows. In Section 2, we explain the concept of polycrystalline symmetry. In Section 3, we relate this kind of symmetry to the $3D$ lattice unit cells identified by Bravais and show that spacetimes with this symmetry gather into four classes. Section 4 describes how to obtain the general forms of the metric for a spacetime with a polycrystalline symmetry. This will allow us to assert that spacetimes belonging to three of these classes are physically meaningless. Section 5 gives the general form of the curvature tensor for the only class of spacetimes with polycrystalline symmetry which may correspond to something real. Finally, Section 6 is devoted to the conclusion and an interpretation of our results.

## 2. Crystalline and polycrystalline symmetries :

We are interested in a characterization of spacetimes whose spacelike hypersurfaces have an Euclidean crystalline structure at a given scale. More specifically, we want to describe the spacetimes with spacelike hypersurfaces having a $3D$ Euclidean crystalline structure which can be generated by more than one distinct Bravais unit cell. This crystalline structure possesses more than one space lattice. The transitions between such lattices are performed through a linear transformation corresponding to a change of the crystallographic coordinate system. The crystalline structure of these spacelike hypersurfaces is invariant under this kind of transformations. The spacelike hypersurfaces having the above invariance property will be said to have a polycrystalline structure. We shall say that a spacetime $V_4$ has a polycrystalline symmetry if to each transformation $A$ preserving the polycrystalline structure of its spacelike hypersurfaces, there corresponds a transformation of $V_4$ into $V_4$, also denoted by $A$, which leaves invariant the metric and the tensor curvature of $V_4$. Such a transformation applies to both static and non-static $V_4$.

To find the general forms of the metric for a spacetime with a polycrystalline symmetry, one needs to describe the structure of $3D$ Euclidean crystals. This description rests on six parameters among which three are vectors $a$, $b$, $c$, and three are scalars $\alpha$, $\beta$, $\gamma$. The first three are concurrent edges of the structure's basic parallelepiped called unit cell, while the last three are the angles made by the corresponding edges, *i.e.* between the pairs $(b, c)$, $(c, a)$ and $(a, b)$, respectively. A crystalline structure can then be seen as the lattice generated by the repetition through space of a unit cell keeping the orientation.

The group of symmetry, or space group, of a crystalline structure is the combination of all possible transformations which leave invariant the structure. This group is the direct product of the unit cell point group, with the group of discrete translations of the structure lattice. The crystalline structure symmetry restricts the number of possible lattices in the $3D$ Euclidean space $E^3$. The lattices realizable in $E^3$ are specified by their unit cells which have been identified by Bravais (see e.g. [2], p. 75).

**3. The fourteen Bravais unit cells :**

The notion of primitive cell is needed to present the Bravais classification. A unit cell is called primitive if it contains no interior lattice points, being otherwise non-primitive. The primitive cells are the most fundamental, since such a cell immediately defines the lattices. However, a non-primitive cell can sometimes exhibits symmetry features not otherwise apparent.

The most symmetric unit cell is the cube. Extending or contracting one of its edges produces the tetragonal cell. Extending or contracting a second edge produces the orthorhombic cell. Changing one of the angles from $\pi/2$ produces the monoclinic cell. Changing a second angle from $\pi/2$ produces the triclinic cell. There exist two further cells which do not fit directly into the preceding scheme. First, extending or contracting a cube uniformly along one of its diagonals produces a rhombohedron. Secondly, changing the angle $\gamma$ of the tetragonal cell from $\pi/2$ to $2\pi/3$ produces a hexagonal cell, so called since this cell generates the primitive hexagonal lattice. It can be proved that the rhombohedral cell is equivalent to a double-centred hexagonal cell. However, it is often convenient to treat this as a different system. Accordingly, the primitive unit cells may be classified into seven standard systems depending on their symmetry properties.

Together with the preceding seven full cells, there are seven other independent non-primitive cells which share their symmetry properties. These are first the cubic, tetragonal, orthorhombic and monoclinic body-centred cells. Secondly, there is the orthorhombic end-centred cell. Finally, there are the cubic and orthorhombic face-centred cells. The primitive, body-, end- and face-centred cells are respectively designated by $P$, $I$, $C$ and $F$. The rhombohedral or its equivalent non-primitive hexagonal cell is identified by $R$. More than one lattice, each one resulting from the juxtaposition of one given Bravais cell, can lie within one another to form a complex crystalline structure.

Five of the Bravais unit cells are simple in the sense that each of these is associated with a unique space lattice. The other nine Bravais unit cells can be used in more than one crystalline lattice of $E^3$. These latter unit cells gather into four classes which we shall identify by words and with the crystallographic international symbols : 1) the rhombohedral and hexagonal cells ($R\bar{3}m$) ; 2) the end-centred

monoclinic ($C2/m$) and orthorhombic ($Cmmm$) cells; 3) the face-centred orthorhombic ($Fmmm$) and cubic ($Fm3m$) cells; 4) the body-centred orthorhombic ($Immm$), tetragonal ($I4/mmm$) and cubic ($Im3m$) cells.

If $a$, $b$, $c$ are the vectors generating the space lattice made with rhombohedral cells, and $a'$, $b'$, $c'$ are the vectors generating the lattice made with hexagonal cells, it is easily shown that the linear transformation $A_1$ between these sets of vectors is given by

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Similarly, the transitions between the space lattices associated with the second, third and fourth classes of unit cells may be expressed, respectively, by the matrices ([2], p. 620)

$$A_2 = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A_3 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

**4. Polycrystalline metrics :**

We determine all the general forms a spacetime metric can take, when the spacetime obeys a polycrystalline symmetry. Let us first assume that the spacetime $V_4$ has a 3 coordinate system $x^i$, $i = 0, 1, 2, 3$, in which $V_4$ can be sliced into a family of spacelike hypersurfaces, $x^0$ being constant over each member of the family. Let $ds$ be the infinitesimal distance between two nearby points of $V_4$. The coordinate differences $dx^i$, $i = 0, 1, 2, 3$, between these points are the components of a vector $dr$ in $V_4$. The squared element of length $ds^2$ of $V_4$ can then be seen as a quadratic form of the vector $dr$. If we designate by $(g_{ij}(x^0, x^1, x^2, x^3))$ the matrix representation of this quadratic form with respect to the coordinate system $x^i$, $i = 0, 1, 2, 3$, then

$$ds^2 = g_{ij} dx^i dx^j.$$

We now consider a change of coordinate system for $V_4$ and designate by $\bar{x}^i$, $i = 0, 1, 2, 3$, its new coordinate system. With respect to $\bar{x}^i$, the square of the element of length of $V_4$ is given by

$$ds^2 = \bar{g}_{ij} d\bar{x}^i d\bar{x}^j,$$

where $\bar{g}_{ij} = \bar{g}_{ij}(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)$. Let us assume that this change of coordinate system leaves invariant the hypersurface decomposition and the time coordinate labels of each hypersurface of the family while at the same time transforming the spacelike coordinates according to one of the matrices $A_k$, $k = 1, 2, 3, 4$. This implies that for each $k = 1, 2, 3, 4$, the coordinate differences of two neighbouring points of $V_4$ are related by

$$(d\bar{x}^0, d\bar{x}^1, d\bar{x}^2, d\bar{x}^3)^T = A_k (dx^0, dx^1, dx^2, dx^3)^T,$$

where

$$A_k = \text{diag}(1, A_k), \qquad k = 1, 2, 3, 4.$$

Since such changes of coordinate system leave unchanged the inherent crystalline structure of the spacelike hypersurfaces, the metric of the corresponding spacetime must be the same in both the old and the new coordinate systems. This means that

$$\bar{g}_{ij}(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3) = g_{ij}(x^0, x^1, x^2, x^3)$$

for $i, j = 0, 1, 2, 3$. Consequently, a spacetime will have a polycrystalline symmetry of the $k^{\text{th}}$ class if and only if its metric $G = (g_{ij})$ satisfies

$$G = A_k G A_k^T. \tag{1}$$

We now apply (1) to the four classes of polycrystalline symmetries identified by the $A_k$. It is an easy task to show that the most general forms of the metric are given by, respectively,

$$G_1 = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad G_2 = \begin{pmatrix} g_{00} & 0 & 0 & g_{03} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g_{03} & 0 & 0 & g_{33} \end{pmatrix}, \quad G_3 = \begin{pmatrix} g_{00} & g_{01} & g_{01} & g_{01} \\ g_{01} & g_{11} & g_{11} & g_{11} \\ g_{01} & g_{11} & g_{11} & g_{11} \\ g_{01} & g_{11} & g_{11} & g_{11} \end{pmatrix},$$

$$G_4 = \begin{pmatrix} g_{00} & 0 & 0 & 0 \\ 0 & -(g_{12}+g_{13}) & g_{12} & g_{13} \\ 0 & g_{12} & -(g_{12}+g_{23}) & g_{23} \\ 0 & g_{13} & g_{23} & -(g_{13}+g_{23}) \end{pmatrix},$$

where the $g_{ij}$ are arbitrary functions of $x^i$, $i = 0, 1, 2, 3$.

According to general relativity, any spacetime is characterized by a metric whose components satisfy the Einstein equations. It is well known that this metric can have local singularities, *i.e.* points where the determinant of its matrix realization vanishes. However, a metric whose determinant is identically zero must be rejected because it would correspond to nothing having a physical sense. Therefore, any spacetime of the first three classes of polycrystalline symmetries are physically impossible. Only spacetimes with polycrystalline symmetry of the fourth class have a chance to describe something real. By determining the general form of the curvature tensor of spacetimes of this fourth class, we shall see in the next section what kind of further restrictions are imposed on these spacetimes.

## 5. Curvature tensor for spacetimes of the fourth class :

Let us characterize the general form of the curvature tensor $R$ of a spacetime $V_4$ belonging to the fourth class of polycrystalline symmetry. The tensor $R$ must be invariant under the change of coordinate system described by the matrix $A_4$. It follows that the algebraically independent components of $R$, at any given point of $V_4$, will also be invariant under the same transformation. This property will then hold for the ten independent components of the Weyl tensor and the ten independent components of the Ricci tensor which form the twenty independent components of the most general form of $R$.

To determine the general form of the Ricci tensor $(R_{ij})$ of a spacetime with the fourth class of polycrystalline symmetry, one can observe that its matrix realization has the same properties as those of the metric. Using the same arguments as those applied to the metric, we thus obtain that the general form of $(R_{ij})$ directly follows from $G_4$, by simply replacing the non trivial $g_{ij}$ with the corresponding $R_{ij}$.

A condition similar to (1) for the Weyl tensor $C$ of a spacetime with the fourth class of polycrystalline symmetry results from the Petrov matrix expression of its ten algebraically independent components ([3], p. 176). To obtain this expression, we use the following correspondence between pairs of tensor indices of $C$ and single Petrov indices :

$$\begin{array}{ll} \text{Tensor indices :} & ij,\, kl = 23,\ 31,\ 12,\ 10,\ 20,\ 30; \\ & \updownarrow \\ \text{Petrov index :} & A, B = \ 1,\ \ 2,\ \ 3,\ \ 4,\ \ 5,\ \ 6; \end{array}$$

The matrix of independent components of $C$ takes a simpler form if, instead of the fully covariant components $C_{ijkl}$, one considers the mixed components $C^{ij}_{kl} \leftrightarrow C^{A}_{B}$. Here, we have $C^{A}_{B} = G^{AC} C_{CB}$, where the matrix $(G^{AC}) = \mathrm{diag}(I_{3\times3}, -I_{3\times3})$, and $I_{3\times3}$ is the $3\times3$ identity matrix. The ten independent components of $C$ are then given by

$$(C^{A}_{B}) = \begin{pmatrix} M & N \\ -N & M \end{pmatrix},$$

where $M = (m_{ij})$ and $N = (n_{ij})$ are symmetric traceless $3\times3$ matrices.

To the coordinate transformation

$$(\bar{x}^0, \bar{x}^1, \bar{x}^2, \bar{x}^3)^T = A_4 (x^0, x^1, x^2, x^3)^T, \tag{2}$$

corresponds a similarity transformation of the matrix $C = (C^{A}_{B})$. Denoting with an overbar the components of the Weyl tensor in the barred coordinate system $\bar{x}^i$, $i = 0, 1, 2, 3$, one obtains ([3], p. 178)

$$\bar{C}^{ij}_{kl} = \sum_{(mn,pd)\leftrightarrow \text{Petrov}} \left( 2 \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial \bar{x}^j}{\partial x^n} \right) C^{mn}_{pq} \left( 2 \frac{\partial x^p}{\partial \bar{x}^k} \frac{\partial x^q}{\partial \bar{x}^l} \right), \tag{3}$$

where the sum is taken only over the pairs $mn$ and $pq$ corresponding to Petrov indices. If the Petrov indices $A$, $B$, $C$, $D$ correspond, respectively, to the pairs of tensor indices $ij$, $kl$, $mn$, $pq$, then (3) is equivalent to

$$\bar{C}^{A}_{B} = S^{A}_{C}\, C^{C}_{D}\, \tilde{S}^{D}_{B}, \tag{4}$$

where

$$S_C^A = 2\frac{\partial \bar{x}^i}{\partial x^m}\frac{\partial \bar{x}^j}{\partial x^n}, A, C = 1, 2,..., 6,$$

and the $\tilde{S}_B^D$ are the components of $\tilde{S}$, the inverse of the matrix $S = (S_C^A)$, which in our case is given by

$$S = 2\,\text{diag}\left[\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\right]. \tag{5}$$

For the Weyl tensor of $V_4$ to be invariant under the coordinate transformation (2), it is necessary that $\bar{C}_B^A = C_B^A, A, B = 1, 2,..., 6$. Equations (4) then becomes

$$C_B^A = S_C^A C_D^C \tilde{S}_B^D$$

which is equivalent to

$$CS = SC. \tag{6}$$

The substitution of (5) into (6) directly leads us to conclude that the general form of the Weyl tensor for a spacetime with the fourth class of polycrystalline symmetry is given by

$$C = \begin{pmatrix} \widehat{M} & \widehat{N} \\ -\widehat{N} & \widehat{M} \end{pmatrix}, \tag{7}$$

where

$$\widehat{M} = \begin{pmatrix} m_{11} & m_{12} & m_{11}+m_{12}+2m_{22} \\ m_{12} & m_{22} & 2m_{11}+m_{12}+m_{22} \\ m_{11}+m_{12}+2m_{22} & 2m_{11}+m_{12}+m_{22} & -(m_{11}+m_{22}) \end{pmatrix},$$

and $\widehat{N}$ has the same form as $\widehat{M}$ with $n_{ij}$ replacing $n_{ij}$. In terms of $C_{kl}^{ij}$, the expression (7) means that all components of $C$ may be written using only six independent components, which can be chosen to be $C_{10}^{23}, C_{20}^{23}, C_{23}^{23}, C_{31}^{23}, C_{20}^{31}, C_{31}^{31}$. It also leads to the following non-standard symmetry relations.

$$C^{23}_{12} = C^{23}_{23} + C^{23}_{31} + 2C^{31}_{31}, \qquad C^{23}_{30} = C^{23}_{10} + C^{23}_{20} + 2C^{31}_{20},$$

$$C^{31}_{12} = 2C^{23}_{23} + C^{23}_{31} + C^{31}_{31}, \qquad C^{31}_{30} = 2C^{23}_{10} + C^{23}_{20} + 2C^{31}_{20},$$

$$C^{12}_{12} = -C^{23}_{23} - C^{31}_{31}, \qquad C^{12}_{30} = -C^{23}_{10} - C^{31}_{20}.$$

Note that the coordinate system used to express the above general form of the Weyl tensor is not necessarily the same as the one used to give the corresponding general forms of the metric and Ricci tensors. However, the set of components we have identified as independent remain independent in all coordinate systems.

## 6. Conclusion and discussion :

We have shown that every spacetime with the fourth class of polycrystalline symmetry has a metric and a Ricci tensor whose general form is essentially given by $G_4$, while the general form of its Weyl tensor is (7). Observe that knowing the general forms of the metric and curvature tensor of this class of spacetimes does not ensure the existence of such spacetimes. However, these general forms may be used as starting points to check if any spacetime of this kind really exist: one has to solve the corresponding Einstein equations to do that.

A more direct result of this paper applies to any spacetime whose spacelike hypersurfaces have a polycrystalline structure of the first three classes. We have shown that, according to general relativity, such spacetimes have no physical reality. This conclusion also applies to the central part of a relatively large region of space with such a crystalline structure. But the existence of any material body requires a non-vanishing matter tensor, and thus of a non-vanishing curvature tensor which in turn is determined by a metric whose components satisfy the Einstein equations. This implies that no object can be created with spatial region having the exact polycrystalline symmetry of any one of the first three classes. The very existence of any material object requires that it be at best imperfectly symmetric in the sense of the first three polycrystalline classes.

We shall now give two examples showing that the preceding conclusion contradicts results from crystallography. Let us first observe that as soon as a spacelike hypersurface has one crystalline structure of one of the first three classes

of polycrystalline symmetry, then it automatically has the other crystalline structure of the same class. Thus, if a crystal has the spatial structure generated by the repetition of the hexagonal unit cell, then it also has the rhombohedral lattice structure. The case just described will precisely be the one considered in our first example. According to crystallography, the atoms of carbon in graphite form a crystalline structure made up from the hexagonal unit cell (see *e.g.* [4], p. 39). Thus, the graphite will also have the rhombohedral crystalline structure. But we have seen that general relativity forbids this kind of polycrystalline symmetry. As a second example of contradiction between general relativity and crystallography, we consider the numerous single crystals of random size forming a piece of metallic copper. Crystallography tells us that these crystals result from the repetition of the face-centred cubic unit cell ([4], p. 41). Therefore, these crystals also have a lattice structure based on the face-centred orthorhombic unit cell. Once again, general relativity does not permit such a polycrystalline symmetry.

As in [1], the application of discrete invariance properties to general relativity has led us to contradictions between two successful theories used to describe our world. Note that we did not show any contradiction between one of these theories and reality itself. It is in fact possible that no physical object with a perfect polycrystalline symmetry of the first three classes really exists. In particular, it is possible that the atoms of carbon of a piece of graphite form a spatial structure which is very close to being a perfect hexagonal lattice without being it exactly. Our results just put forward the fact that one cannot apply simultaneously general relativity and crystallography to the same domain without intrinsic contradictions.

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