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Integer Solutions of Some Exponential Diophantine Equations

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Abstract:

In this paper, we search for non-negative integer solutions to the exponential Diophantine equations. We discussed theorems for their integer solutions.

Keywords: Catalan conjecture, Exponential Diophantine equation, Integer solutions, Diophantine Equations.

Introduction:

Mathematicians look for and use examples to figure out new conjectures; they settle reality or duplicity of conjectures by numerical verification. Numbers are however unbounded as human arrangement seems to be confined, so number hypothesis and its particular subfields will keep energizing the personalities of mathematicians for a very extensive time frame [1-3]. An outstanding Diophantine equation is an equation where the types are integers. Various Diophantine equations are solved by Catalan conjecture [4-7].

In [8], solution of exponential Diophantine equation involving Jarasandha numbers is acquired utilizing Catalan conjecture. In this correspondence, we look for non-negative integer solutions to a special exponential Diophantine equation utilizing Catalan conjecture.

Catalan's Conjecture:

The Diophantine equation $a^x + b^y = 1$ has unique integer solution with min $\{a, b, x, y\} > 1$. The solution (a, b, x, y) is (3, 2, 2, 3). This was proved by Mihailescu in 2004.

Method of Analysis:

Section - A

The exponential Diophantine equation under consideration is

$$1^x + 3^y = z^2 \tag{1}$$

Theorem 1: (0,1, 2) is the solution of the Exponential Diophantine equations $1^x + 3^y = z^2$, where x, y & z are non-negative integers.

Proof: Suppose $x = 0$, then $1^x + 3^y = z^2$ becomes $z^2 - 1 = 3^y$

Let $z - 1 = 3^{u_1}$, (2)

where u_1 are non-negative integers.

Then $z + 1 = 3^{y-u_1}$ (3)

Using (2) & (3), we get $3^{y-2u_1} - 1 = 2$

$\Rightarrow \quad y = 1, u_1 = 0$

If $x = 0$ then $y = 1$, so that $z = 2$

Hence (0, 1, 2) is the solution of $1^x + 3^y = z^2$

Theorem 2: The exponential Diophantine equation has no solution.

Proof: Suppose $y = 0$ & write $1^x + 3^y = z^2$ as $1^x + 1 = z^2$

Let $z - 1 = 1^{u_1}$, (4)

where u_1 is a non-negative integer.

Utilizing the same method in case 1, we get $u_1 = 0$ & $1^x = 3$. Which is also impossible for positive values of x, so that $y \neq 0$.

The exponential Diophantine equation has no solution.

Theorem 3: (1, 1, 2) is the solution of the Exponential Diophantine equation $1^x + 3^y = z^2$, where x, y & z are non-negative integers.

Proof: Suppose $x \geq 1$, rewrite (1) as $3^y = z - 1^x z + 1^x$

Let $z - 1^x = 3^{u_1}$, (5)

where u_1 is the non-negative integer.

Then $z + 1^x = 3^{y-u_1}$ (6)

Using (5) & (6), $z + 1^x - z + 1^x = 3^{y-u_1} - 3^{u_1}$

$\Rightarrow \qquad u_1 = 0$

then $2^x = 3^y - 1$.

If $x = 1$ then $y = 1$.

So that, $z = 2$

Hence (1, 1, 5) is the solution $1^x + 3^y = z^2$.

Then, in general, (0, 1, 2) & (1, 1, 5) are the solutions of the Exponential Diophantine equation $1^x + 3^y = z^2$.

Section - B

The exponential Diophantine equation under consideration is

$$4^x + 3^y = z^2$$

Theorem 4: (0, 1, 2) is the solution of the Exponential Diophantine equation $4^x + 3^y = z^2$, where x, y & z are negative integers.

Proof: Consider the equation

$$4^x + 3^y = z^2 \tag{7}$$

Suppose $x = 0$, then $1^x + 3^y = z^2$ becomes $z^2 - 1 = 3^y$

Let $z - 1 = 3^{u_2}$, (8)

where u are non-negative integers.

Then $z + 1 = 3^{y-u_2}$ (9)

Using (2) & (3), we get $3^{y-2u_2} - 1 = 2$

$\Rightarrow \quad y = 1, u_2 = 0$

If $x = 0$ then $y = 1$, so that $z = 2$.

Hence (0, 1, 2) is the solution of $4^x + 3^y = z^2$

Theorem 5: The exponential Diophantine equations has no solution.

Proof: Suppose $y = 0$ and write (7) as $4^x + 1 = z^2$

Let $z - 1 = 1^{u_2}$, (10)

where u_2 is a non-negative integer.

Utilizing the same method in case 1, we get $u_2 = 0$ and $4^x = 3$ which is also impossible for positive values of x, so that $y \neq 0$.

Theorem 6: (1, 1, 2) is the solution of the Exponential Diophantine equation $1^x + 3^y = z^2$, where x, y & z are non-negative integers.

Proof: Suppose $x \geq 1$, rewrite (7) as $3^y = z - 2^x z + 2^x$

Let $z - 2^x = 3^{u_2}$, (11)

where u_2 is the non-negative integer.

Then $z + 2^x = 3^{y-u_2}$ (12)

Using (11) & (12), $z + 2^x - z + 2^x = 3^{y-u_2} - 3^{u_2}$

$\Rightarrow \qquad u_2 = 0$

then $2.2^x = 3^y - 1$

If $x = 2$ then $y = 2$.

So that, $\quad z = 5$.

Hence (2, 2, 5) is the solution of $4^x + 3^y = z^2$.

Then, in general, (0, 1, 2) & (2, 2, 5) are the solutions of the Exponential Diophantine equation $4^x + 3^y = z^2$.

Section - C

The exponential Diophantine equation under consideration is

$$9^x + 3^y = z^2$$

Theorem 7: (0, 1, 2) is the solutions of the Exponential Diophantine equation $9^x + 3^y = z^2$ where x, y & z are negative integers.

Proof: Consider the equation

$$9^x + 3^y = z^2 \qquad (13)$$

Suppose $x = 0$, then $1^x + 3^y = z^2$ becomes $Z^2 - 1 = 3^y$

Let $z - 1 = 3^{u_3}$, (14)

where u_3 is non-negative integer.

Then $z + 1 = 3^{y-u_3}$ (15)

Using (14) & (15), we get $3^{y-2u_3} - 1 = 2$

$\Rightarrow \quad y = 1, u_3 = 0.$

If $x = 0$ then $y = 1$, so that $z = 2$.

Hence (0, 1, 2) is the solution of $1^x + 3^y = z^2$

Theorem 8: The exponential Diophantine equations has no solution.

Proof: Suppose $y = 0$ and write (13) as $9^x + 1 = z^2$

Let $z - 1 = 9^{u_3}$, (16)

where u_3 is a non-negative integer.

Utilizing the same method in case 1, we get $u_3 = 0$ and $9^x = 3$ which is also impossible for positive values of x, so that $y \neq 0$.

Theorem 9: (1, 3, 6) is the solution of the Exponential Diophantine equation $4^x + 3^y = z^2$ where x, y & z are non-negative integers.

Proof: Suppose $x \geq 1$, $9^x + 3^y = z^2$ as $3^y = z \ - 3^x z + 3^x$

Let $z - 3^x = 3^{u_3}$, (17)

where u_3 is the non-negative integer.

Then $z + 3^x = 3^{y-u_3}$ (18)

Using (17) & (18), $z + 3^x - z + 3^x = 3^{y-u_3} - 3^{u_3}$

$\Rightarrow \quad u_3 = 1$

then $2.3 = 3^{u_3} (3^{y-2u_3} - 1)$

If $x = 1$ then $y = 3$.

So that, $\quad z = 6$

Hence (1, 3, 6) is the solution of $9^x + 3^y = z^2$.

Then, in general, (0, 1, 2) & (1, 3, 6) are the solutions of the Exponential Diophantine equation $9^x + 3^y = z^2$.

Section - D

The exponential Diophantine equation under consideration is

$$16^x + 3^y = z^2$$

Theorem 10: (0, 1, 2) is the solution of the Exponential Diophantine equation $16^x + 3^y = z^2$ where x, y & z are negative integers.

Proof: Consider the equation

$$16^x + 3^y = z^2 \tag{19}$$

Suppose $x = 0$, then $1^x + 3^y = z^2$ becomes $Z^2 - 1 = 3^y$

Let $z - 1 = 3^{u_4}$, (20)

where u_4 are non-negative integers.

Then $Z + 1 = 3^{y-u_4}$ (21)

Using (14) & (15), we get $3^{y-2u} - 1 = 2$

$\Rightarrow \quad y = 1, u_4 = 0$

If $x = 0$ then $y = 1$, so that $z = 2$.

Hence (0, 1, 2) is the solution of $16^x + 3^y = z^2$.

Theorem 11: The exponential Diophantine equation has no solution.

Proof: Suppose $y = 0$ and write (13) as $16^x + 1 = z^2$

Let $z - 1 = 16^{u_4}$ (22)

Where u_4 is a non-negative integer.

Utilizing the same method in case 1, we get $u_4 = 0$ and $16^x = 3$ which is also impossible for positive values of x, so that $y \neq 0$.

Theorem 12: (1, 2, 5) is the solution of the Exponential Diophantine equation $16^x + 3^y = z^2$ where x, y & z are non-negative integers.

Proof: Suppose $x \geq 1$, $16^x + 3^y = z^2$ as $3^y = z - 4^x z + 4^x$

Let $z - 4^x = 3^{u_4}$, (23)

where u_4 is the non-negative integer.

Then $z + 4^x = 3^{y-u_4}$ (24)

Using (23) & (24), $z + 4^x - z + 4^x = 3^{y-u_4} - 3^{u_4}$

$\Rightarrow \qquad u_4 = 0$

then $8^x = 3^y - 1$.

If $x = 1$ then $y = 2$.

So that, $z = 5$.

Hence (1, 2, 5) is the solution of $16^x + 3^y = z^2$.

Then, in general, (0, 1, 2) & (1, 2, 5) are the solutions of the Exponential Diophantine equation $16^x + 3^y = z^2$.

Section - E

The exponential Diophantine equation under consideration is

$$3^x + 2^y = z^2 \tag{25}$$

Theorem 13: (0, 3, 3) is the solution of the Exponential Diophantine equations $3^x + 2^y = z^2$, where x, y & z are non-negative integers.

Proof: Suppose $x = 0$, then $3^x + 2^y = z^2$ becomes $Z^2 - 1 = 2^y$

Let $z - 1 = 2^{u_1}$ (26)

where u_1 is non-negative integer.

Then $Z + 1 = 2^{y-u_1}$ (27)

Using (26) & (27), we get $2^{y-2u_1} - 1 = 1$

$\Rightarrow \qquad y = 3, u_1 = 1$

If $x = 0$ then $y = 3$, so that $z = 3$.

Hence (0, 3, 3) is the solution of $3^x + 2^y = z^2$.

Theorem 14: (1, 0, 2) is the solution of the exponential Diophantine equation $3^x + 2^y = z^2$ where x, y & z are non-negative integers.

Proof: Suppose $y = 0$ & then $3^x + 2^y = z^2$ becomes $z^2 - 1 = 3^x$ (28)

Let $z - 1 = 3^{u_1}$, (29)

where u_1 is a non-negative integer.

Then $z + 1 = 3^{x-u_1}$ (30)

Using (29) & (30), we get $3^{x-2u_1} - 1 = 2$

$\Rightarrow \qquad u_1 = 0$

If $y = 0$ then $x = 1$, so that $z = 2$.

Hence (1, 0, 2) is the solution of $3^x + 2^y = z^2$.

Theorem 15: The Exponential Diophantine equation $3^x + 2^y = z^2$ has no solution.

Proof: Suppose $x \geq 1$, rewrite (1) as $z^2 - 3 = 2^y$

Could not find the values of x & y.

It has no solution.

Section - F

The exponential Diophantine equation under consideration is

$$6^x + 2^y = z^2$$

Theorem 16: (0, 3, 3) is the solution of the Exponential Diophantine equation $6^x + 2^y = z^2$, where x, y & z are negative integers.

Proof: Consider the equation

$$6^x + 2^y = z^2 \tag{31}$$

Suppose $x = 0$, then $6^x + 2^y = z^2$ becomes $Z^2 - 1 = 2^y$

Let $z - 1 = 2^{u_2}$, (32)

where u_2 is non-negative integer.

Then $Z + 1 = 2^{y-u_2}$ (33)

Using (32) & (33), we get $2^{y-2u_2} - 1 = 1$

$\Rightarrow \quad y = 3, u_2 = 1$

If $x = 0$ then $y = 3$, so that $z = 3$.

Hence (0, 3, 3) is the solution of $13^x + 2^y = z^2$.

Theorem 17: The Exponential Diophantine equation $6^x + 2^y = z^2$ has no solution.

Proof: Suppose $y = 0$ and write (31) as $6^x + 1 = z^2$

Let $z - 1 = 6^{u_2}$, (34)

where u_2 is a non-negative integer.

Make use of the same method in case 1, we get $u_2 = 0$ and $6^{x-2u_2} = 3$ which is also impossible for positive values of x, so that $y \neq 0$.

Theorem 18: (2, 6, 10) is the solution of the Exponential Diophantine equation $6^x + 2^y = z^2$, where x, y & z are non-negative integers.

Proof: Suppose $x \geq 1$.

If $x = 2$ then rewrite (7) as $2^y = z^2 - 6^2$

Let $z - 6 = 2^{u_2}$, (35)

where u_2 is the non-negative integer.

Then $z + 6 = 2^{y-u_2}$ (36)

Using (36) & (35), $z + 6 - z + 6 = 2^{y-u_2} - 2^{u_2}$

$\Rightarrow \quad u_2 = 2$

then, $2^{y-2u_2} - 1 = 3$

If $x = 2$ then $y = 6$, so that $z = 10$.

Hence (2, 6, 10) is the solution of (31).

Then, in general, (0, 1, 2) & (2, 2, 5) are the solutions of the Exponential Diophantine equation $4^x + 3^y = z^2$.

Section - G

The exponential Diophantine equation under consideration is

$$9^x + 2^y = z^2$$

Theorem 19: (0, 3, 3) is the solution of the Exponential Diophantine equation $9^x + 2^y = z^2$ where x, y & z are negative integers.

Proof: Consider the equation

$$9^x + 2^y = z^2 \tag{37}$$

Suppose $x = 0$, then $9^x + 2^y = z^2$ becomes $z^2 - 1 = 2^y$

Let $z - 1 = 2^{u_3}$, (38)

where u_3 is non-negative integer.

Then $z + 1 = 2^{y-u_3}$ (39)

Using (38) & (39), we get $2^{y-2u_3} - 1 = 1$

$\Rightarrow \qquad y = 3, u_3 = 1$

If $x = 0$ then $y = 3$, so that $z = 3$.

Hence (0, 3, 3) is the solution of $9^x + 2^y = z^2$.

Theorem 20: The Exponential Diophantine equation $9^x + 2^y = z^2$ has no solution.

Proof: Suppose $y = 0$ and write (37) as $9^x + 1 = z^2$

Let $z - 1 = 9^{u_3}$, (40)

where u_3 is a non-negative integer.

Make use of the same method in case 1, we get $u_3 = 0$ and $9^{x-2u_3} = 3$ which is also impossible for positive values of x, so that $y \neq 0$.

Theorem 21: (1, 4, 5) is the solution of the Exponential Diophantine equation $9^x + 2^y = z^2$, where x, y & z are negative integers.

Proof: Suppose $x \geq 1$, $9^x + 2^y = z^2$ as $z^2 - 3^2 = 2^y$

Let $z - 3 = 2^{u_3}$, (41)

where u_3 is non-negative integer.

Then $z + 3 = 2^{y-u_3}$ (42)

Using (42) & (41), $z + 3 - z + 3 = 2^{y-u_3} - 2^{u_3}$

$\Rightarrow \qquad u_3 = 1$

then $2.3 = 2^{u_3}(2^{y-2u_3} - 1)$

If $x = 1$ then $y = 4$, so that $z = 5$.

Hence (1, 4, 5) is the solution of $9^x + 2^y = z^2$.

Then, in general, (0, 3, 3) & (1, 4, 5) are the solutions of the Exponential Diophantine equation $9^x + 3^y = z^2$.

Section - H

The exponential Diophantine equation under consideration is

$$12^x + 2^y = z^2$$

Theorem 22: (0, 3, 3) is the solution of the Exponential Diophantine equation $12^x + 2^y = z^2$, where x, y & z are negative integers.

Proof: Consider the equation

$$12^x + 2^y = z^2 \tag{43}$$

Suppose $x = 0$, then $12^x + 2^y = z^2$ becomes $z^2 - 1 = 2^y$

Let $z - 1 = 2^{u_4}$, (44)

where u_4 is non-negative integer.

Then $z + 1 = 2^{y-u_4}$ (45)

Using (44) & (45), we get $2^{y-2u_4} - 1 = 1$

$\Rightarrow$ $y = 3$, $u_4 = 1$

If $x = 0$ then $y = 3$, so that $z = 3$.

Hence (0, 3, 3) is the solution of $12^x + 2^y = z^2$.

Theorem 23: The Exponential Diophantine equation $12^x + 2^y = z^2$ has no solution.

Proof: Suppose $y = 0$ and write (43) as $12^x + 1 = z^2$

Let $z - 1 = 12^{u_4}$ (46)

where u_4 is a non-negative integer.

Make use of the same method in case 1, we get $u_4 = 0$ and $12^x = 3$ which is also impossible for positive values of x, so that $y \neq 0$.

Theorem 24: (2, 8, 20) is the solution of the Exponential Diophantine equation $12^x + 2^y = z^2$, where x, y & z are negative integers.

Proof: Suppose $x \geq 1$, if $x = 2$ then $12^x + 2^y = z^2$ becomes $2^y = z^2 - 12^2$

Let $z - 12 = 2^{u_4}$, (47)

where u_4 is the non-negative integer.

Then $z + 12 = 2^{y-u_4}$ (48)

Using (47) & (48), $z + 12 - z + 12 = 2^{y-u_4} - 2^{u_4}$

$$u_4 = 3$$

then $2^{y-2u_4} - 1 = 3$.

If $x = 2$ then $y = 8$, so that $z = 20$.

Hence (2, 8, 20) is the solution of $12^x + 2^y = z^2$.

Then, in general, (0, 3, 3) & (2, 8, 20) are the solutions of the Exponential Diophantine equation $12^x + 2^y = z^2$.

Conclusion:

The exponential Diophantine equation has non-zero unique integer solutions, which we have shown in this work. In order to draw a conclusion, one could look for other equations using different numbers.

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