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Some Applications of Antimagic Graphs in Cryptology

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Abstract:

This paper presents some Cryptology applications of Antimagic labelled graphs. The graphs used are namely Complete Bipartite, Path and Wheel. The resulting Cipher-texts are very cryptic and encryption and decryption algorithms are presented for all these cases including one independently discovered algorithm to label the Wheel graphs with Antimagic graph labelling.

Keywords: Antimagic graph labelling, encryption, decryption, Plain-text, Cipher-text, lexicographic order, Antimagic Path Matrix, Antimagic Label Matrix, Antimagic Complete Bipartite Matrix.

1. Introduction:

Antimagic labelling is found in **Hartsfield and Ringel [1]. Deepa, Maheswari and Balaji [2]** have used graph labellings for encryption and decryption. The Inner

Magic and Inner Antimagic graph labellings were used in Cryptography as well as Cryptology application in **Krishnaa [3]** which also discussed certain relevant possibilities and aspects of graph labellings as the graphs grow larger in size. In **Krishnaa [4],** Complete Bipartite graphs have been shown to be Antimagic using *lexicographic order* of the edge labels. A *lexicographic order* is like the order in a lexicon (dictionary) in which the word 'care' comes before 'cart'. **Gurjar and Krishnaa [5]** have used Antimagic Path for Cryptography as well as Cryptology application providing encryption and decryption algorithms. **Vasuki, Shobana and Roopa [6]** have presented encryption of data using Antimagic labelling and Hill Cipher. **Jegan, Vijaykumar and Thirusangu [7]** have given encryption of a word using certain variations of Antimagic labelling. **Krishnaa [8]** has used Lattices of Discrete Mathematics for Cryptology applications.

Section 2 presents the main results with subsection 2.1 presenting the algorithm to label the Wheel graph with Antimagic graph labelling and the Cryptology application. Subsection 2.2 presents the Antimagic Path with 2 kinds of Cryptology applications by using findings obtained in **Gurjar and Krishnaa [5]:** (A) using 3 kinds of Cipher-texts (B) using the Antimagic Path Matrix. Subsection 2.3 presents the Cryptology application using the Complete Bipartite graph.

2. Main Results:

2.1 Wheel:

Other algorithms for Antimagic Wheels also may exist and we begin by presenting the algorithm to label the Wheel with Antimagic labelling which has been independently discovered for this work.

Antimagic graph labelling is one in which the edges are labelled with 1, 2, …, e such that the sum of the edge labels at each vertex is distinct (e is the total number of edges in the graph)

Algorithm Antimagic Wheel W_n (n is the number of outer as well as inner edges):

1. Label the outer n edges with integers 1, 2, 3, …, n clockwise.
2. Label the inner n edges with the integers $(n + 1)$, $(n + 2)$, $(n + 3)$, …, $2n$, starting at the inner edge lying between outer edges labelled with 1 and n.

3. Resulting labelling is Antimagic. Certain patterns observed are as follows:

   W_4 :

   Smallest Antimagic label being $x = 2n + 1 = 9$ (adjacent outer edges labelled with 1 and 2) and the Antimagic vertex label $y = 2n + 2$, immediate left to x.

   $W_5 : x = 2n, y = 2n + 2$
   $W_7 : x = 2n - 2, y = 2n + 2$
   $W_8 : x = 2n - 3, y = 2n + 2$, etc.

From x, the Antimagic vertex labels form an Arithmetic Progression (A.P.) with common difference $d = 3$.

W_6 is not Antimagic as per this labelling scheme since there is repetition of one vertex label so a different scheme of labelling is presented which is Antimagic as shown in the **Figure 2**.

Even though there is use of numbers but due to the graphs being pictorial or diagrammatical, the behaviour of numbers in graphs is not as predictable - this aspect has been discussed in **Krishnaa [3]** as well.

Illustration :

Encryption :

For handling longer Plain-texts, letters of Plain-text are assigned to the edges instead of to the vertices.
Plain-text : GRAPHCYCLE

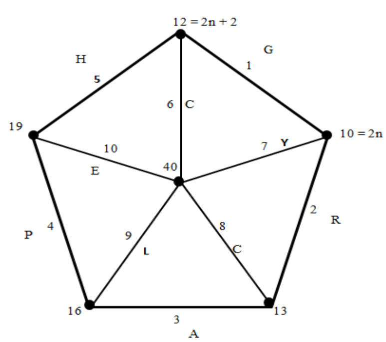

Figure 1 : Antimagic Wheel W_5

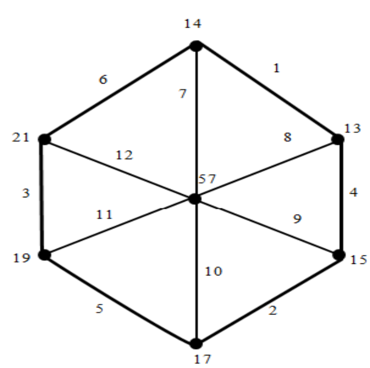

Figure 2 : Antimagic Wheel W_6

1. Assign G, R, A, P, H to the outer edges labelled with 1, 2, 3, 4, 5 respectively.
2. Assign C, Y, C, L, E to the inner edges labelled with 6, 7, 8, 9, 10 respectively.

   As shown in **Figure 1**.
3. Make the Cipher-text by taking the *average* of the 2 Antimagic labels of the 2 end-vertices of the edge assigned with a letter of the Plain-text as follows:

   Outer circle:

   $G = (12 + 10)/2 = 22/2 = 11$ or $G + 11 = R$

   $R = (10 + 13)/2 = 23/2$, take it as $22/2 = 11$ or $R + 11 = C$

   $A = (13 + 16)/2 = 29/2$, take it as $28/2 = 14$ or $A + 14 = O$

   $P = (19 + 16)/2 = 35/2$, take it as $34/2 = 17$ or $P + 17 = G$

   $H = (12 + 19)/2 = 31/2$, take it as $30/2 = 15$ or $H + 15 = W$

   Inner circle:

   $C = (40 + 12)/2 = 52/2 = 26$ which is same as 0 in *modulo addition* 26 since there are 26 alphabets in English. So, $C + 0 = C$

   $Y = (40 + 10)/2 = 50/2 = 25$ or $Y + 25 = X$

   $C = (40 + 13)/2 = 53/2$, take it as $52/2 = 26 = 0$ or $C + 0 = C$

   $L = (40+ 16)/2 = 56/2 = 28 = 2$ (as per *addition modulo* 26) or $L + 2 = N$

   $E = (40 + 19)/2 = 59/2$, take it as $58/2 = 29 = 3$ or $E + 3 = H$
4. Therefore the Cipher-text is **RCOGWCXCNH**.

   The letters can also be represented with each letter as a 2-digit number yielding a numeric Cipher-text: **18031507230324031408**.
5. Key : 11111417150025000203 (each 2-digit number is the *average* obtained clockwise in outer circle then in the inner circle).
6. Send the Key, Cipher-text to the receiver.

Decryption:

Subtract each 2 digit number of the Key from each letter of the Cipher-text left to right as follows:

$R - 11 = G$, $C - 11 = R$, $O - 14 = A$, $G - 17 = P$, $W - 15 = H$, $C - 0 = C$, $X - 25 = Y$, $C - 0 = C$, $N - 2 = L$, $H - 3 = E$ therefore yielding the original Plain-text: GRAPHCYCLE.

2.2 Path:

(A) Using 3 kinds of Cipher-texts:

As per Gurjar and Krishnaa (2021), 5 schemes have been given for Antimagic Path. For illustration purpose here we are taking 3 schemes of these. Scheme (c) has been modified somewhat in this work to demonstrate that the *reverse* operation to that used in encryption is applied in decryption. We proceed as follows:

Illustration:

Encryption:

1. First of all assign letters of the Plain-text GRAPH to the edges of Path with 5 vertices from left to right as G, R, A, P, H as shown in the **Figure 3**.

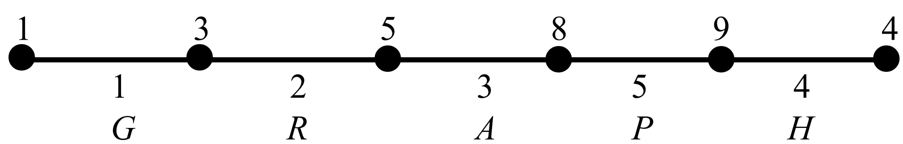

Figure 3: Antimagic Path

$$\begin{array}{c} \\ \mathbf{1} \\ \mathbf{3} \\ \mathbf{5} \\ \mathbf{8} \\ \mathbf{9} \\ \mathbf{4} \end{array} \begin{array}{c} \begin{array}{cccccc} \mathbf{1} & \mathbf{3} & \mathbf{5} & \mathbf{8} & \mathbf{9} & \mathbf{4} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 & 0 & 4 \\ 0 & 0 & 0 & 0 & 4 & 0 \end{bmatrix} \end{array}$$

Antimagic Label Matrix

2. **Scheme (a):** adding the right Antimagic vertex label to the letter of the Plain-Text.
   Scheme (b): adding the left Antimagic vertex label to the letter of the Plain-Text.
   Scheme (c): subtracting the *absolute difference* of the left and right Antimagic end-vertex labels corresponding to the end-vertices of the edge to which the letter of the Plain-Text is assigned.

3. As per Scheme (a), the Cipher-text 1 is JWIXL
   As per Scheme (b), the Cipher-text 2 is HUFXQ
   As per Scheme (c), the Cipher-text 3 is EPXOC

4. Set the Key1 for Scheme (a) as $R+$ (adding the Right vertex label)
   Set the Key2 for Scheme (b) as $L+$ (adding the Left vertex label)
   Set the Key3 for Scheme (c) as AD - (subtracting the absolute difference AD)

5. Send the Antimagic Label Matrix, Cipher-text1, Cipher-text2, Cipher-text3 and their respective keys Key1, Key2 and Key3 to the receiver.

Decryption:

Decryption uses the reverse operation to that of Encryption. Using the Antimagic Label Matrix we can draw the Antimagic Path graph and we proceed as follows:

Cipher-text1:

$J-3=G$, $W-5=R$, $I-8=A$, $Y-9=P$, $L-4=H$ (subtracting the right vertex label) thus yielding GRAPH.

Cipher-text2:

$H-1=G$, $U-3=R$, $f-5=A$, $X-8=P$, $Q-9=H$ (subtracting the left vertex label) thus yielding GRAPH.

Cipher-text 3:

$E+2=G$, $P+2=R$, $X+3=A$, $O+1=P$, $C+5=H$ (adding the result obtained of the *absolute difference*) thus yielding GRAPH.

Therefore, in all the 3 cases the original Plain-text GRAPH is obtained.

(B) Using the Antimagic Path Matrix:

Illustration:

Encryption:

1. Make the Antimagic Path Matrix as follows :

$$\begin{array}{cc} & \begin{array}{cccccc}\mathbf{1}&\mathbf{3}&\mathbf{5}&\mathbf{8}&\mathbf{9}&\mathbf{4}\end{array} \\ \begin{array}{c}\mathbf{1}\\\mathbf{3}\\\mathbf{5}\\\mathbf{8}\\\mathbf{9}\\\mathbf{4}\end{array} & \begin{bmatrix}0&07&0&0&0&0\\07&0&18&0&0&0\\0&18&0&01&0&0\\0&0&01&0&16&0\\0&0&0&16&0&08\\0&0&0&0&08&0\end{bmatrix}\end{array}$$

Antimagic Path Matrix

Instead of the forward position number of the letters i.e., (3 for *C*), the reverse position number of the letter also can be used e.g., 3 for *X*) to enhance the hiding capacity of the letters.

2. Send the Antimagic Path Matrix to the receiver.

Decryption:

The position numbers of the letters represent the letters of the Plain-text along the diagonal in the upper triangular matrix or the lower triangular matrix.

2.3 Complete Bipartite Graph:

In the Complete Bipartite Graph $K_{2,3}$, the vertices may represent offices/ cities and the edges represent the routes to go from one vertex to any other vertex. To go from vertex v_1 to v_5, the allowed secret routes are assigned certain numbers which are obtained by adding the Antimagic vertex labels to the edge labels on the paths from v_1 to v_5. For instance, as per the **Figure 4**, the allowed v_1 to v_5 secret routes are as follows:

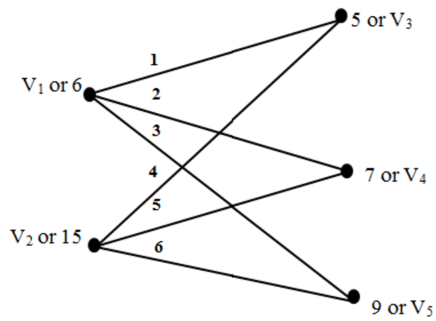

Figure 4: Antimagic Complete Bipartite Graph $K_{2,3}$

(i) $6(v_1) + 2 + 7(v_4) + 5 + 15(v_2) + 6 + 9 = 50$ so this valid route may be written as v_1v_5, 50 or 061550, where v_1 has the antimagic vertex label of 6 and v_5 has 15 and 50 is the unique valid route number.

(ii) $6(v_1) + 2 + 7(v_4) + 5 + 15(v_2) + 6 = 41$; v_1v_5, 41 or 061541

(iii) $6(v_1) + 3 + 9(v_5) = 18$; v_1v_5,18 or 061518

Illustration:

Encryption:

1. Make the Antimagic Complete Bipartite Matrix as shown below.

$$\begin{array}{cc} & \begin{array}{ccc} \mathbf{5} & \mathbf{7} & \mathbf{9} \end{array} \\ \begin{array}{c} \mathbf{6} \\ \mathbf{15} \end{array} & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{array}$$

Antimagic Complete Bipartite Matrix

2. Send the Antimagic Complete Bipartite Matrix and the route to check for validity : v_1v_5, 35 to the receiver.

Decryption:

1. Draw the Complete Bipartite Graph $K_{2,3}$ which is Antimagic from the Antimagic Complete Bipartite Matrix.

2. Calculate the various plausible numbers of the various valid routes. In this case since $35 \neq 41$, $35 \neq 50$ and $35 \neq 18$ so 35 is not the valid secret route number.

Conclusion:

This work has presented encryption and decryption algorithms for Path, Wheel and Complete Bipartite graphs for Cryptology applications including the Antimagic labelling algorithm for Wheel which is independently arrived at. The resulting Cipher-texts can be alphabetical as well as numeric and are highly cryptic impossible or too difficult to crack since it is not quite possible to even guess the structure or the labelling/labelling scheme of the graphs used for these applications. Therefore a highly secure data transfer can be secured using these methods.

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