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A Study of Exponential Functions of Operators such as Projection

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Abstract :

In this paper, projection will be defined in a linear space. As Simmons G.F. "Introduction to topology and modern analysis." Projection is a linear operator A on a linear space such that $A^2 = A$, we defined e^A and leads to hyperbolic functions defined on A such as sinhA, coshA, sechA, tanhA, and some interesting theorems.

Keywords : Linear space, Projection operator, Hyperbolic functions, Hilbert Space.

1. Introduction :

Projection Operator and Hyperbolic Functions :

Definition : Let L be a linear space[6]. Let A be a projection on L, then A is a linear transformation from L into L such that, $A^2 = A$[7].

Define,

$$e^A = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \text{-----------}$$

$$= I + A + \frac{A}{2!} + \frac{A}{3!} + \frac{A}{4!} + \text{-----------}$$

$$= I + A\left(1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \text{-----------}\right)$$

$$= I + A(e - 1)$$

Also,

$$e^{2A} = (e^A)^2 = [I + A(e - 1)]^2$$

$$= I + 2A(e - 1) + A^2(e - 1)^2$$

$$= I + 2A(e - 1) + A(e - 1)^2$$

$$= I + A(2e - 2 + e^2 - 2e + 1)$$

$$= I + A(e^2 - 1)$$

Clearly, e^A is not a projection.

$$e^{3A} = (e^A)^3 = (e^A)^2 \,.\, e^A$$

$$= [I + A(e^2 - 1)]\,[I + A(e - 1)]$$

$$= I + A(e - 1) + A(e^2 - 1) + A(e^2 - 1)(e - 1)$$

$$= I + A(e - 1 + e^2 - 1 + e^3 - e^2 - e + 1)$$

$$= I + A(e^3 - 1)$$

Assume,

$$e^{nA} = I + A(e^n - 1)$$

Then,

$$e^{(n+1)A} = e^{nA} . e^{A}$$

$$= [I + A(e^{n} - 1)]\,[I + A(e - 1)]$$

$$= I + A(e - 1) + A(e^{n} - 1) + A(e^{n} - 1)(e - 1)$$

$$= I + A(e - 1 + e^{n} - 1 + e^{n+1} - e^{n} - e + 1)$$

$$= I + A(e^{n+1} - 1)$$

Hence by induction for any positive integers n,

$$e^{nA} = I + A(e^{n} - 1)$$

If we put $n = 0$, then

$$e^{0A} = I + A(e^{0} - 1)$$

$$= I$$

Thus,

$e^{0} = I$, where 0 is zero operator.

Also,

$$[I + A(e^{-n} - 1)]\,[I + A(e^{n} - 1)]$$

$$= I + A(e^{n} - 1) + A(e^{-n} - 1) + A(e^{-n} - 1)(e^{n} - 1)$$

$$= I + A(e^{n} - 1 + e^{-n} - 1 + e^{0} - e^{-n} - e^{n} + 1)$$

$$= I + A(e^{0} - 1)$$

$$= I$$

So we can write inverse function

$$e^{-nA} = I + A(e^{-n} - 1)$$

Thus if n is a negative integer, then also,

$$e^{nA} = I + A(e^n - 1)$$

Hence for any integer n,

$$e^{nA} = I + A(e^n - 1)$$

Let B be another projection, then

$$e^B = I + B(e - 1)$$

Then we find that,

$$e^A \,.\, e^B = [I + A(e - 1)]\,[I + B(e - 1)]$$

$$= I + B(e - 1) + A(e - 1) + AB(e - 1)^2$$

$$= I + (A + B)(e - 1) + AB(e - 1)^2$$

We need A+B to be a projection i.e,

$$(A + B)^2 = A + B$$

$$\Rightarrow \quad A^2 + 2AB + B^2 = A + B$$

Assuming A, B commute this is possible, when $AB = 0$.

Then,

$$e^A \,.\, e^B = e^{A+B}$$

Also,

$$e^{nA} \,.\, e^{mB} = [I + A(e^n - 1)]\,[I + A(e^m - 1)]$$

$$= I + A(e^m - 1) + A(e^n - 1) + A(e^n - 1)(e^m - 1)$$

$$= I + A(e^m - 1 + e^n - 1 + e^{n+m} - e^n - e^m + 1)$$

$$= I + A(e^{n+m} - 1)$$

$$= e^{(m+n)A}$$

I, identity operator is also a projection and

$$e^{I} = I + I(e - 1) = eI$$

$I - A$ is also a projection. Hence

$$e^{I-A} = I + (I - A)(e - 1)$$

$$= I + eI - I - eA + A$$

$$= eI + A(1 - e)$$

$$e^{A} . e^{I-A} = e^{A+I-A} = e^{I} = eI$$

Also,

$$e^{A} . e^{I-A} = [I + A(e - 1)] [eI + A(1 - e)]$$

$$= eI + A(1 - e) + A(e - 1)e - A^2(e - 1)^2$$

$$= eI + A(1 - e) + A(e - 1)e - A(e - 1)^2, \text{ As } A^2 = A$$

$$= eI + A(1 - e + e^2 - e - e^2 + 2e - 1)$$

$$= eI + A.0$$

$$= eI$$

2. We know that in a Hilbert space H, if P is a projection, then $P^2 = P$, $P^* = P$ and $P \geq 0$.

Theorem (2.I) : If A is a projection operator on a Hilbert space H^8. Then e^A is non-singular.

Proof : We need to show that e^A or $I + A(e - 1)$ is one-to-one and onto as a mapping of H into itself. First we show that it is one-to-one.

Let $\quad e^{A}x = 0$

$\Rightarrow \quad [I + A(e-1)]x = 0$

$\Rightarrow \quad x + (e-1)Ax = 0$

Applying A on both sides,

$$Ax + (e-1)A^2x = 0$$

$\Rightarrow \quad Ax + (e-1)Ax = 0,\quad$ since A is projection operator i.e. $A^2 = A$

$\Rightarrow \quad eAx = 0$

$\Rightarrow \quad Ax = 0$

Hence

$\Rightarrow \quad x = 0$

We next show that e^A is onto, i.e. its range is H. Let M be the range of e^A.

Let $x \in L$ then,

$$x + \left(\frac{1}{e} - 1\right)Ax \in L$$

Let $\quad y = x + \left(\frac{1}{e} - 1\right)Ax \in L$

Then,

$$Ay = Ax + \left(\frac{1}{e} - 1\right)A^2x = Ax + \left(\frac{1}{e} - 1\right)Ax = \frac{1}{e}Ax$$

Consider $\quad y + (e-1)Ay$ or $e^{A}y$

Then,

$$y + (e-1)\,Ay = x + \left(\frac{1}{e} - 1\right)Ax + \frac{e-1}{e}Ax$$

$$= x + \left(\frac{1}{e} - 1\right) Ax + \left(1 - \frac{1}{e}\right) Ax = x$$

Thus, $x = e^A y$ i.e. $x \in M$

Hence, $L \subseteq M \subseteq L$ or $L = M$

Thus e^A is one-to-one and onto i.e. non singular.

3. Definition :

Next, we are going to define hyperbolic function involving a projection operator A.

Let $\cos hA = I + A(\cos h1 - 1)$

$$= I + A\left(\frac{e + e^{-1}}{2} - 1\right)$$

And $\sin hA = A \sin h1 = A\left(\frac{e - e^{-1}}{2}\right)$

Hence, $\cos hA + \sin hA = I + A(e - 1) = e^A$

and $\cos hA - \sin hA = I + A(e^{-1} - 1)$

Hence, $\cos h^2 A - \sin h^2 A = [I + A(e - 1)]\,[I + A(e^{-1} - 1)]$

$$= I + A(e^{-1} - 1) + A(e - 1) + A(e - 1)(e^{-1} - 1)$$

$$= I + A(e^{-1} - 1 + e - 1 + 1 - e - e^{-1} + 1) = I$$

This is analogous to $\cos h^2 x - \sin h^2 x = 1$ when x is a real number.

We have seen before that,

$$e^{nA} = I + A(e^n - 1)$$

and $e^{-nA} = I + A(e^{-n} - 1)$

Adding,

$$2\cosh nA = 2I + A(e^{n} + e^{-n} - 2)$$

$$= 2I + A(2\cosh n - 2)$$

$$\Rightarrow \quad \cosh nA = I + A(\cosh n - 1) \tag{3.1}$$

Also subtraction gives,

$$e^{nA} - e^{-nA} = A(e^{n} - e^{-n})$$

$$\Rightarrow \quad 2A\sinh n = A.2\sinh n$$

$$\Rightarrow \quad \sinh nA = A\sinh n \tag{3.2}$$

Putting $n = -1$ in (3.1) and (3.2),

$$\cosh(-A) = I + A[\cosh(-1) - 1]$$

$$= I + A(\cosh 1 - 1)$$

$$= \cosh A$$

And $\sinh(-A) = A\sinh(-1) = -A\sinh 1 = -\sinh A$

Putting $n = 2$ in (3.2),

$$\sinh 2A = A\sinh 2 = A2\sinh 1\cosh 1 \tag{3.3}$$

Also $2\sinh A\cosh A = 2A\sinh 1(I + A(\cosh 1 - 1))$

$$= 2A\sinh 1 + 2A\sinh 1(\cosh 1 - 1)$$

$$= 2A\sinh 1\cosh 1 \tag{3.4}$$

From (3.3) and (3.4),

$$\sinh 2A = 2\sinh A\cosh A$$

Theorem (3.I) : We show that,

$$\cos h2A = \cos h^2A + \sin h^2A$$

Proof : Putting $n = 2$ in (3.1),

$$\cos h2A = I + A(\cos h2 - 1) \tag{3.5}$$

Now,

$$\cos h^2A + \sin h^2A = [I + A(\cos h1\text{-}1)]^2 + (A\sin h1)^2$$

$$= I + A^2(\cos h1\text{-}1)^2 + 2A(\cos h1\text{-}1) + A^2\sin h^2 1$$

$$= I + A[\cos h^2 1 + 1 - 2\cos h1 + 2\cos h1 - 2 + \sin h^2 1]$$

$$= I + A(\cos h^2 1 + \sin h^2 1 + 1 - 2)$$

$$= I + A(\cos h2 - 1) \tag{3.6}$$

From (3.5) and (3.6),

$$\cos h2A = \cos h^2A + \sin h^2A$$

Theorem (3.II) : Let us assume $AB = 0$.

Then we show that $\sin h(A + B) = \sin hA \cos hB + \cos hA \sin hB$

Proof : We have

$$\sin h(A + B) = (A + B)\sin h1$$

Also,

$$\sin hA \cos hB + \cos hA \sin hB$$

$$= (A\sin h1)[I + B(\cos h1\text{-}1)] + [I + A(\cos h1\text{-}1)]\, B\sin h1$$

$$= A\sin h1 + B\sin h1$$

$$= (A + B)\sin h1$$

$$= \sin h(A + B)$$

Theorem (3.III) : We show $\cos h(A+B) = \cos hA\,.\cos hB - \sin hA\,.\sin hB$

Proof : We have,

$$\cos hA \cos hB - \sin hA \sin hB = [I + A(\cos h1\text{-}1)\text{-}1]\,[I + B(\cos h1\text{-}1)\text{-}1]$$

$$- (A\sin h1)(B\sin h1)$$

$$= I + B(\cos h1\text{-}1) + A(\cos h1\text{-}1)$$

$$= I + (A+B)(\cos h1\text{-}1)$$

$$= \cos h(A+B).$$

4. Definition : Let us consider $I + A(\sec hn - 1)$.

$$\cos hnA[I + A(\sec hn - 1)] = [I + A(\cos hn - 1)]\,[I + A(\sec hn - 1)]$$

$$= I + A(\sec hn - 1) + A(\cos hn - 1)$$

$$+ A^2(\cos hn - 1)(\sec hn - 1)$$

$$= I + A(\sec hn - 1 + \cos hn - 1$$

$$+ 1 - \cos hn - \sec hn + 1 = I$$

Hence define,

$$\sec hnA = I + A(\sec hn - 1)$$

Define, $\tan hnA = \sin hnA\,.\sec hnA$

Theorem (4.I) : $\tan hnA = A\tan hn$

Proof : Now,

$$\sin hnA\,.\sec hnA = [A\sin hn][I + A(\sec hn - 1)]$$

$$= A\sin hn + A^2\sin hn(\sec hn - 1)$$

$$= A\sinh n(1 + \operatorname{sech} n - 1)$$

$$= A\sinh n\operatorname{sech} n$$

$$= A\tanh n$$

Hence, $\tanh nA = A\tanh n$

Theorem (4.II) : $\operatorname{sech}^2 A + \tanh^2 A = I$

Proof : $\operatorname{sech}^2 A + \tanh^2 A$

$$= I + A(\operatorname{sech}1 - 1)^2 + A^2\tanh^2 1$$

$$= I + A^2(\operatorname{sech}1 - 1)^2 + 2A(\operatorname{sech}1 - 1) + A^2\tanh^2 1$$

$$= I + A[\operatorname{sech}^2 1 + 1 - 2\operatorname{sech}1 + 2\operatorname{sech}1 - 2 + \tanh^2 1]$$

$$= I + A[\operatorname{sech}^2 1 + \tanh^2 1 + 1 - 2] = I + A(1 + 1 - 2) = I$$

Theorem (4.III) : $\tanh 2A(I + \tanh^2 A) = 2\tanh A$

Proof : We have,

$$\tanh 2A = A\tanh 2$$

And, $I + \tanh^2 A = I + A^2\tanh^2 1$

$$= I + A\tanh^2 1$$

Hence, $\text{L.H.S} = A\tanh 2(I + A\tanh^2 1)$

$$= \frac{A.2\tanh 1}{1 + \tanh^2 1}(1 + A\tanh^2 1)$$

$$= \frac{A.2\tanh 1 + A.2\tanh^3 1}{1 + \tanh^2 1}$$

$$= \frac{2A\tan h1(1 + \tan h^2 1)}{1 + \tan h^2 1}$$

$$= 2A\tan h1$$

$$= 2\tan hA$$

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