

The Mathematics Education

ISSN 0047-6269

Volume - LIX, No. 2, June 2025

Journal website: <https://internationaljournalsiwan.com>

ORCID Link: <https://orcid.org/0009-0006-7467-6080>

International Impact Factor: 7.75 <https://impactfactorservice.com/home/journal/2295>

Google Scholar: <https://scholar.google.com/citations?hl=en&user=UOfM8B4AAAAJ>

Refereed and Peer-Reviewed Quarterly Journal



Idempotent & Nilpotent Operators and Matrices in Bicomplex Space

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(Received: April 12, 2025; Accepted: May 5, 2025; Published Online: June 30, 2025)

Abstract:

This paper explores idempotent and nilpotent operators in bicomplex spaces, focusing on their properties and behavior. We define idempotent and nilpotent matrices in this framework and derive related results. Several theorems are presented to establish conditions for the existence and behavior of bicomplex idempotent and nilpotent operators and bicomplex idempotent matrices.

2020 Mathematics Subject Classification: Primary 15A04, 15A30; Secondary 30G35.

Keywords and Phrases: Bicomplex Numbers, Conjugation, Vector Space and Linear transformation.

1. Introduction:

The theory of bicomplex numbers is a central focus of contemporary mathematical research, with significant progress in recent years. Numerous authors (see [1, 9, 10, 11, 12]) have advanced the field, exploring diverse perspectives to elucidate their properties and establish a consistent framework for the multivariate theory of complex numbers. Recently, researchers studying matrices and linear operators (see [2, 3, 4, 5, 8]) over various algebraic systems have made extensive contributions to mathematics. Bicomplex numbers, introduced by Segre, extend the concept of complex numbers and form a commutative ring with zero divisors. Their properties find applications in functional analysis, quantum mechanics, and signal processing.

2. Preliminaries and Notations:

This section provides an introduction to bicomplex numbers and explores their key properties. It highlights several essential findings related to bicomplex numbers.

Bicomplex numbers: Bicomplex numbers are an extension of complex numbers, defined as:

$$\xi = u_1 + i_1 u_2 + i_2 u_3 + i_1 i_2 u_4,$$

where u_1, u_2, u_3 and u_4 are real numbers with $i_1 i_2 = i_2 i_1, i_1^2 = i_2^2 = -1$.

The collection of all bicomplex numbers is represented by $\mathbb{C}_2$ and is referred to as the bicomplex space. For simplicity, $\mathbb{C}_1$ stands for the set of complex numbers, and $\mathbb{C}_0$ indicates the set of real numbers. The bicomplex space $\mathbb{C}_2$ can be characterized in two distinct ways:

$$\mathbb{C}_2 := \{u_1 + i_1 u_2 + i_2 u_3 + i_1 i_2 u_4 : u_1, u_2, u_3, u_4 \in \mathbb{C}_0\}, \text{ and}$$

$$\mathbb{C}_2 := \{z_1 + i_2 z_2 : z_1, z_2 \in \mathbb{C}_1\}.$$

The set C_2 contains zero-divisors, which makes it an algebra over C_1 rather than a field. Within C_2 , there are exactly four idempotent elements: 0, 1, e_1 , e_2 , where e_1 and e_2 are two nontrivial idempotent elements, specified as follows:

$$e_1 := \frac{(1+i_1i_2)}{2} \text{ and } e_2 := \frac{(1-i_1i_2)}{2}.$$

These elements stand out due to their orthogonality ($e_1e_2 = e_2e_1 = 0$) and the fact that they add up to 1 ($e_1 + e_2 = 1$).

(1) Also, $e_1^n = e_1$ and $e_2^n = e_2$; $n \in \mathbb{N}$.

Idempotent Representation and Equality Condition of Bicomplex Numbers:

Every bicomplex number ξ has unique idempotent representation as complex combination of e_1 and e_2 as follows

$$\xi = z_1 + i_2z_2 = (z_1 - i_1z_2)e_1 + (z_1 + i_1z_2)e_2$$

The complex numbers $(z_1 - i_1z_2)$ and $(z_1 + i_1z_2)$ are called idempotent component of ξ , and are denoted by ξ^- and ξ^+ respectively (cf. Srivastava [11]). Thus, the bicomplex number can be written as

$$\xi = \xi^- e_1 + \xi^+ e_2.$$

Furthermore, for two bicomplex numbers $\xi, \eta \in C_2$.

$$\xi = \eta \text{ if and only if } \xi^- = \eta^-, \xi^+ = \eta^+.$$

That is, bicomplex numbers are equal if and only if their corresponding idempotent components coincide.

Definition 2.1: ([5], [Definition 1.4]): A bicomplex matrix of order $m \times n$ is written as $A = [\xi_{ij}]_{m \times n}$, $\xi_{ij} \in C_2$ with each element $\xi_{ij} \in C_2$. The collection of all such bicomplex matrices is denoted $C_2^{m \times n}$, defined as:

$$(2) \quad C_2^{m \times n} =: \{ [\xi_{ij}] : \xi_{ij} \in C_2; i = 1, 2, \dots, m, j = 1, 2, \dots, n \}.$$

With usual matrix addition and scalar multiplication, the set $C_2^{m \times n}$ forms a vector space over the field C_1 . The dimension of $C_2^{m \times n}$ over C_1 is immediately given by

$$(3) \quad \dim (\mathbb{C}_2^{m \times n})(\mathbb{C}_1) = 2mn.$$

Furthermore, each bicomplex matrices A uniquely decomposes as $A = [\xi_{ij}]_{m \times n} \in \mathbb{C}_2^{m \times n}$ can be decomposed uniquely as

$$(4) \quad A = e_1 A^- + e_2 A^+,$$

where $A^- = [\xi_{ij}^-]_{m \times n}$, $A^+ = [\xi_{ij}^+]_{m \times n}$ are complex matrices.

Remark 2.2: Analogous to the concept of equality of two bicomplex numbers, two bicomplex matrices $A = e_1 A^- + e_2 A^+$, $B = e_1 B^- + e_2 B^+ \in \mathbb{C}_2^{m \times n}$ are equal if and only if their idempotent component matrices are equal. That is,

$$(5) \quad A = B \text{ if and only if } A^- = B^- \text{ and } A^+ = B^+,$$

and the product, sum of two bicomplex matrices and bicomplex scalar product are decomposed as follows:

$$(6) \quad A \cdot B = e_1(A^- \cdot B^-) + e_2(A^+ \cdot B^+).$$

$$(7) \quad (A + B) = e_1(A^- + B^-) + e_2(A^+ + B^+).$$

$$(8) \quad \xi \cdot A = e_1(\xi A^-) + e_2(\xi A^+); \forall \xi \in \mathbb{C}_2.$$

Remark 2.3: ([4], [Remark 3.1]): To streamline notation, denote the set of all $\mathbb{C}_1$ -linear maps from $\mathbb{C}_1^n$ to $\mathbb{C}_1^m$ by L_1^{nm} , and set of all $\mathbb{C}_1$ -linear maps from $\mathbb{C}_2^n$ to $\mathbb{C}_2^m$ by L_2^{nm} . Both are vector spaces over $\mathbb{C}_1$, with dimensions:

$$(9) \quad \dim(L_1^{nm}) = mn \text{ and } \dim(L_2^{nm}) = \dim \mathbb{C}_2^n \cdot \dim \mathbb{C}_2^m = 2n \cdot 2m = 4mn.$$

Since $\mathbb{C}_1$ is a field, $L_1^{nm} \cong \mathbb{C}_1^{m \times n}$. However $\mathbb{C}_2$ is a not field, $L_2^{nm} \not\cong \mathbb{C}_2^{m \times n}$. Instead, $\mathbb{C}_2^{m \times n}$ is a proper subspace of L_2^{nm} , leading to the next definition.

Definition 2.4: ([4], [Definitions 3.2, 4.1]): For any given $T_1, T_2 \in L_1^{nm}$, we can define a map $T: \mathbb{C}_2^n \rightarrow \mathbb{C}_2^m$ by the following rule:

$$T(\xi_1, \xi_2, \dots, \xi_n) = e_1 \cdot T_1(\xi_1^-, \xi_2^-, \dots, \xi_n^-) + e_2 \cdot T_2(\xi_1^+, \xi_2^+, \dots, \xi_n^+).$$

Clearly T is a $\mathbb{C}_1$ -linear map. T can also be represented by $e_1 T_1 + e_2 T_2$. Thus the set of all such linear maps is the idempotent product $L_1^{nm} \times_e L_1^{nm}$, i.e., we have

$$(10) \quad L_1^{nm} \times_e L_1^{nm} =: \{e_1 T_1 + e_2 T_2 \in L_2^{nm} : T_1, T_2 \in L_1^{nm}\}.$$

For convenience, the set of all such type of $T = e_1 T_1 + e_2 T_2 : \mathbb{C}_2^n \rightarrow \mathbb{C}_2^n$ linear operators is denoted by $L_1^n \times_e L_1^n$. The idempotent product $L_1^{nm} \times_e L_1^{nm}$ is a subspace of L_2^{nm} over the field $\mathbb{C}_1$. This indicates directly that $L_1^{nm} \times_e L_1^{nm}$ has dimension $2mn$. That is

$$(11) \quad \dim(L_1^{nm} \times_e L_1^{nm}(\mathbb{C}_1)) = 2mn.$$

Since $\mathbb{C}_2^{m \times n}$ and $L_1^{nm} \times_e L_1^{nm}$ have same dimensions over $\mathbb{C}_1$, they are isomorphic. Hence, the matrix expression for $T = e_1 T_1 + e_2 T_2$ is defined using the ordered bases $\mathcal{B}_1$ for $\mathbb{C}_1^n$, and $\mathcal{B}_2$ for $\mathbb{C}_1^m$ as follows:

$$(12) \quad [T]_{\mathcal{B}_2}^{\mathcal{B}_1} =: e_1 [T_1]_{\mathcal{B}_2}^{\mathcal{B}_1} + e_2 [T_2]_{\mathcal{B}_2}^{\mathcal{B}_1}.$$

Here, $[T_1]_{\mathcal{B}_2}^{\mathcal{B}_1}$ and $[T_2]_{\mathcal{B}_2}^{\mathcal{B}_1}$ are matrices of T_1 and T_2 for bases $\mathcal{B}_1$ and $\mathcal{B}_2$. If $\mathbb{C}_1^n = \mathbb{C}_1^m$, the matrix representation of $T = e_1 T_1 + e_2 T_2$ with respect to basis $\mathcal{B}$ for $\mathbb{C}_1^n$ is simplified to $[T]_{\mathcal{B}}$ from $[T]_{\mathcal{B}}^{\mathcal{B}}$. Thus, it follows:

$$(13) \quad [T]_{\mathcal{B}} = e_1 [T_1]_{\mathcal{B}} + e_2 [T_2]_{\mathcal{B}}.$$

Proposition 2.5: ([4], [Proposition 3.3]): Let $T, S \in L_1^{nm} \times_e L_1^{nm}$ be any elements such that $T = e_1 T_1 + e_2 T_2$ and $S = e_1 S_1 + e_2 S_2$. Then, we have

$$(1) \quad T + S = e_1(T_1 + S_1) + e_2(T_2 + S_2).$$

$$(2) \quad \alpha T = e_1(\alpha T_1) + e_2(\alpha T_2); \quad \alpha \in \mathbb{C}_1.$$

Theorem 2.6: ([5], [Theorem 2.7]): A linear operator $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ is singular if and only if either T_1 is singular or T_2 is singular.

Previously, [4] introduced the “Idempotent method” for matrix representation a linear map of the form $T = e_1 T_1 + e_2 T_2 : \mathbb{C}_2^n \rightarrow \mathbb{C}_2^m$. This method provides a systematic approach to establishing a one-to-one correspondence between bicomplex matrices $A = [\xi_{ij}]_{n \times n}$ and the linear operator's $T = e_1 T_1 + e_2 T_2$ on finite dimensional vector space $\mathbb{C}_2^n$. This method helps analyze specific classes of matrices and operators in bicomplex spaces, offering a valuable approach for further study. For a detailed discussion on the Idempotent Method, see [4]. With

this foundation in place, we examine idempotent and nilpotent operators and idempotent and nilpotent matrices in bicomplex spaces, which offer unique insights into the structure of bicomplex linear algebra.

Theorem 2.7: ([4], [Theorem 3.4]): Let $T = e_1 T_1 + e_2 T_2$, $S = e_1 S_1 + e_2 S_2$ be any two elements of $L_1^{nm} \times_e L_1^{nm}$. Then, we have

- (1) $T = 0$, if and only if $T_1 = 0$, $T_2 = 0$.
- (2) $T = S$, if and only if $T_1 = S_1$, $T_2 = S_2$.
- (3) $S \circ T = e_1(S_1 \circ T_1) + e_2(S_2 \circ T_2)$, wherever composition defined.

Anjali [4], stated Theorem 2.7 and we build upon this by extending the concept to the case where $T^n = 0 \forall n \in \mathbb{N}$; accordingly, we propose the following theorems.

Theorem 2.8: Let $T = e_1 T_1 + e_2 T_2$ be a elements of $L_1^{nm} \times_e L_1^{nm}$. Then,

$$T^n = e_1 \underbrace{(T_1 \circ T_1 \circ T_1 \dots T_1)}_{n \text{ times}} + e_2 \underbrace{(T_2 \circ T_2 \circ T_2 \dots T_2)}_{n \text{ times}}$$

$$\text{Or } T^n = e_1 T_1^n + e_2 T_2^n; \forall n \in \mathbb{N}$$

Proof: To prove that $T^n = e_1 T_1^n + e_2 T_2^n$, for all $n \in \mathbb{N}$, using the principle of mathematical induction.

Case 1: For $n = 1$, we have

$$T^1 = e_1 T_1^1 + e_2 T_2^1$$

Clearly, the statement holds.

Assume that the property holds for $n = k$, that is

$$T^k = e_1 T_1^k + e_2 T_2^k$$

We need to show that it holds for $n = k + 1$, that is

$$T^{k+1} = e_1 T_1^{k+1} + e_2 T_2^{k+1}$$

Since

$$T^{k+1} = T^k \circ T$$

We substitute T^k with its assumed form:

$$\begin{aligned} T^{k+1} &= [e_1 T_1^k + e_2 T_2^k] \circ [e_1 T_1 + e_2 T_2] \\ &= e_1(T_1^k \circ T_1) + e_2(T_2^k \circ T_2) \quad \{\text{by Theorem 2.7}\} \\ &= [e_1 \underbrace{(T_1 \circ T_1 \circ T_1 \dots T_1)}_{k+1 \text{ times}} + e_2 \underbrace{(T_2 \circ T_2 \circ T_2 \dots T_2)}_{k+1 \text{ times}}] \\ &= e_1 T_1^{k+1} + e_2 T_2^{k+1}. \end{aligned}$$

Using the principle of mathematical induction, the result holds for every natural number n , i.e.

$$T^n = e_1 T_1^n + e_2 T_2^n.$$

This proof holds for any linear operator $T \in L_1^n \times_e L_1^n$.

Thus the theorem is proved.

Theorem 2.9: Let $T = e_1 T_1 + e_2 T_2$, $S = e_1 S_1 + e_2 S_2$ be any two elements of $L_1^{nm} \times_e L_1^{nm}$. Then, we have

$$(1) \quad T^k = 0, \text{ if and only if } T_1^k = 0, T_2^k = 0.$$

$$(2) \quad T^k = S^k, \text{ if and only if } T_1^k = S_1^k, T_2^k = S_2^k.$$

Proof: (1) We need to prove that for any element $T = e_1 T_1 + e_2 T_2 \in L_1^{nm} \times_e L_1^{nm}$, $T^k = 0$ if and only if $T_1^k = 0, T_2^k = 0$.

Suppose,

$$T^k = 0$$

$$\Leftrightarrow e_1 T_1^k + e_2 T_2^k = 0 \quad \{\text{by Theorem 2.8}\}$$

$$\Leftrightarrow T_1^k = 0 \ \& \ T_2^k = 0 \quad \{\text{as } T^k \text{ is L.T. \& by part (1) of Theorem 2.7}\}.$$

(2) We need to prove that for any two elements $T = e_1 T_1 + e_2 T_2, S = e_1 S_1 + e_2 S_2 \in L_1^{nm} \times_e L_1^{nm}$, the equality $T^k = S^k \Leftrightarrow T_1^k = S_1^k, T_2^k = S_2^k$ for some $k \in \mathbb{N}$.

Suppose,

$$T^k = S^k, \text{ for some } k \in \mathbb{N}$$

$$\Leftrightarrow e_1 T_1^k + e_2 T_2^k = e_1 S_1^k + e_2 S_2^k, \text{ for some } k \in \mathbb{N} \quad \{\text{by Theorem 2.8}\}$$

$$\Leftrightarrow T_1^k = S_1^k \text{ and } T_2^k = S_2^k, \text{ for some } k \in \mathbb{N} \quad \{\text{as } T^k, S^k \text{ are L.T. \& by part (2) of Theorem 2.7}\}.$$

Thus the theorem is proved.

3. Bicomplex Nilpotent Operator and Nilpotent Matrices:

In this section, we define bicomplex nilpotent operators and explore related results. For convenience, we introduce the terms $\mathbb{C}_2$ -nilpotent operators and $\mathbb{C}_2$ -nilpotent matrices to specifically refer to nilpotent operators and matrices in bicomplex spaces, respectively.

Definition 3.1: $\mathbb{C}_2$ -nilpotent operator: A linear operator $T \in L_1^{nm} \times_e L_1^{nm}$ is said to be a $\mathbb{C}_2$ -nilpotent operator if $T^n = 0$ for some positive integer n . The smallest such n is called the index of T .

Definition 3.2: $\mathbb{C}_2$ -nilpotent matrix: A matrix $A = e_1 A^- + e_2 A^+ \in \mathbb{C}_2^{m \times n}$ is said to be a $\mathbb{C}_2$ -nilpotent matrix if there exists a positive integer n such that $A^n = 0$. The smallest such n is called the index matrix A .

Theorem 3.3: A linear operator $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ is a $\mathbb{C}_2$ -nilpotent operator if and only if T_1 and T_2 are nilpotent operators.

Proof: Suppose T is a nilpotent operator. Then, there exists a natural number k such that

$$T^k = 0 \text{ or } (e_1 T_1 + e_2 T_2)^k = 0$$

$$\Rightarrow T_1^k = 0 \text{ and } T_2^k = 0 \quad \{\text{by Theorem 2.9}\}$$

$$\Rightarrow T_1 \text{ and } T_2 \text{ will be nilpotent operators.} \quad \{\text{by Definition 3.2}\}$$

Conversely: Let $T_1, T_2 \in L_1^{nm}$ be two nilpotent operators. Then there exists natural numbers k_1, k_2 such that

$$T_1^{k_1} = 0 \text{ and } T_2^{k_2} = 0.$$

This gives that

$$(14) \quad T_1^l = 0 \text{ and } T_2^l = 0 \quad \forall l \in \mathbb{N}; l \geq k_1, k_2.$$

From Theorem 2.8 and let $l = \max(k_1, k_2)$, then

$$\begin{aligned} (T)^l &= (e_1 T_1 + e_2 T_2)^l \\ &= e_1 (T_1)^l + e_2 (T_2)^l \\ &= e_1 0 + e_2 0 \quad \{\text{as } l \geq k_1, k_2 \text{ and by Equation 14}\} \\ &= 0. \end{aligned}$$

Thus, we have a natural number l such that $T^l = 0$. Hence, T will be a nilpotent operator, as required. Thus, the proof of the theorem is complete.

Theorem 3.4: Let $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ be a $\mathbb{C}_2$ -nilpotent operator and let $\mathcal{B}_1$ be the ordered basis for $\mathbb{C}_1^n$ such that $[T_1]_{\mathcal{B}_1} = A^-$, and $[T_2]_{\mathcal{B}_1} = A^+$ if and only if $A = e_1 A^- + e_2 A^+$ is $\mathbb{C}_2$ -nilpotent matrix.

Proof: Suppose $T = T_2 \in L_1^n \times_e L_1^n$ is a $\mathbb{C}_2$ -nilpotent operator. We use Definition 2.4 and Theorem 3.3, we have

T_1 and T_2 are nilpotent operators

$$\Leftrightarrow \exists n_1, n_2 \in \mathbb{N} \text{ such that } T_1^{n_1} = 0 \text{ and } T_2^{n_2} = 0$$

$$\Leftrightarrow \exists \text{ basis } \mathcal{B}_1 \text{ for } \mathbb{C}_1^n \text{ such that } ([T_1]_{\mathcal{B}_1})^{n_1} = 0 \text{ and } ([T_2]_{\mathcal{B}_1})^{n_2} = 0 \text{ are nilpotent matrices}$$

$$\Leftrightarrow e_1 ([T_1]_{\mathcal{B}_1})^n + e_2 ([T_2]_{\mathcal{B}_1})^n = e_1 (A^-)^n + e_2 (A^+)^n = 0$$

$$\Leftrightarrow ([e_1 T_1 + e_2 T_2]_{\mathcal{B}_1})^n = A^n = 0 \quad \{\because e_1^n = e_1, \& e_2^n = e_2; n \in \mathbb{N}\}$$

Thus $A = [e_1 T_1 + e_2 T_2]_{\mathcal{B}_1}$ is a $\mathbb{C}_2$ -nilpotent matrix. Thus the proof of the theorem is complete.

Theorem 3.5: Let $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ be a $\mathbb{C}_2$ -nilpotent operator. Then T_1 and T_2 are singular.

Proof: Suppose T is a $\mathbb{C}_2$ -nilpotent operator. Then, using ([6], Theorem 3.2.4), ([7], Theorem 1, p.n. 590), and Theorem 3.3, we have

- T_1 and T_2 are nilpotent operator
- $\Rightarrow$ All eigenvalue of T_1 and T_2 are zero
- $\Rightarrow (T_1 - 0I)$ is singular and $(T_2 - 0I)$ is singular
- $\Rightarrow T_1$ and T_2 are singular.

Hence T_1 and T_2 are singular, as required. Thus, the proof of the theorem is complete.

The converse of Theorem 3.4 is not true, as seen in the given example.

Example 3.6: Suppose $T_1(z_1, z_2, z_3) = (z_3 + z_2, z_3, 0)$ and $T_2(w_1, w_2, w_3) = (w_1, 0, w_3)$. It is easy to see that T_1 and T_2 are singular operators. For T_1 , we find $T_1^2 = T_1(T_1(z_1, z_2, z_3)) = T_1(z_3 + z_2, z_3, 0) = (z_3 + 0, 0, 0)$, and $T_1^3 = T_1(T_1^2(z_1, z_2, z_3)) = T_1(z_3 + 0, 0, 0) = (0, 0, 0)$. So, $T_1^3 = 0$, the operator T_1 is nilpotent with index 3 because $T_1^3 = 0$, but $T_1^2 \neq 0$. On the other hand, for all $n \geq 1$ we have $T_2^n = T_2 \neq 0$, which shows that T_2 is not nilpotent. Hence, by Theorem 3.3, it follows that T is not nilpotent.

Theorem 3.7: Let $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ be a $\mathbb{C}_2$ -nilpotent operator and let T_1 and T_2 be two nilpotent operators of index k_1 and k_2 respectively. Then T is a $\mathbb{C}_2$ -nilpotent operator of the index $\max(k_1, k_2)$ and vice versa.

Proof: Suppose T_1 and T_2 are nilpotent operators of index k_1 and k_2 respectively. Then

$$T_1^{k_1} = 0, T_1^{k_1-1} \neq 0 \text{ and } T_2^{k_2} = 0, T_2^{k_2-1} \neq 0.$$

Case 1: If $k_1 \leq k_2$. Then, we have

$$T_1^{k_2} = 0$$

Now,

$$\begin{aligned} T^{k_2} &= (e_1 T_1 + e_2 T_2)^{k_2} \\ &= e_1 T_1^{k_2} + e_2 T_2^{k_2} \\ &= 0. \end{aligned} \quad \{ \text{as } T_1^{k_2} = 0 \text{ and } T_2^{k_2} = 0 \}$$

and
$$T^{k_2-1} = (e_1 T_1 + e_2 T_2)^{k_2-1} = e_1 T_1^{k_2-1} + e_2 T_2^{k_2-1} \neq 0$$

 {as $T_2^{k_2-1} \neq 0$ and by Theorem 2.8}

Therefore T is a $\mathbb{C}_2$ -nilpotent operator of index k_2 .

Case 2: If $k_2 < k_1$. Then we can easily prove that as previous T is a $\mathbb{C}_2$ -nilpotent operator of index k_1 . Hence T will be the $\mathbb{C}_2$ -nilpotent operator of index $\max(k_1, k_2)$.

Conversely: Suppose T is a $\mathbb{C}_2$ -nilpotent operator of index k such that

$$T^k = (e_1 T_1 + e_2 T_2)^k = 0.$$

Now, using Theorem 3.3 we have T_1 and T_2 are nilpotent. There exist natural numbers k_1 and k_2 such that

$$T_1^{k_1} = 0, T_1^{k_1-1} \neq 0 \text{ and } T_2^{k_2} = 0, T_2^{k_2-1} \neq 0.$$

Case 1: If $k_1 \leq k_2$. Then

$$(T_1)^{k_2} = 0$$

Thus we have,

$$T^{k_2} = e_1 (T_1)^{k_2} + e_2 (T_2)^{k_2} = 0$$

and

$$T^{k_2-1} = e_1 (T_1)^{k_2-1} + e_2 (T_2)^{k_2-1} \neq 0 \quad (\text{as } T_2^{k_2-1} \neq 0)$$

Therefore T is a $\mathbb{C}_2$ -nilpotent operator of index k_2 .

Case 2: If $k_2 < k_1$. Then we have

$$(T_2)^{k_1} = 0$$

Thus
$$T^{k_1} = e_1 (T_1)^{k_1} + e_2 (T_2)^{k_1} = 0$$

and

$$T^{k_1-1} = e_1 (T_1)^{k_1-1} + e_2 (T_2)^{k_1-1} \neq 0 \quad (\text{as } T_1^{k_1-1} \neq 0).$$

Therefore T is a $\mathbb{C}_2$ -nilpotent operator of index k_1 .

Since if $k_1 \leq k_2$ and $k_2 < k_1$, then T is a $\mathbb{C}_2$ -nilpotent operator of index k_2 and k_1 respectively.

Hence T is a $\mathbb{C}_2$ -nilpotent operator of index $\text{Max}(k_1, k_2) = k$.

The following Corollary 3.8 is immediate consequence of Theorem 3.7.

Corollary 3.8: Suppose $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ is a $\mathbb{C}_2$ -nilpotent operator of index m . Then at least one nilpotent operator T_1 and T_2 will be of index m .

4. Bicomplex Idempotent Operator and Idempotent Matrices:

In this section, we define bicomplex idempotent operators and matrices and explore their related results. For convenience, we introduce the terms $\mathbb{C}_2$ -idempotent operators and $\mathbb{C}_2$ -idempotent matrices to refer specifically to idempotent operators and matrices in bicomplex spaces.

Definition 4.1: $\mathbb{C}_2$ -idempotent operator: A linear operator $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ is said to be $\mathbb{C}_2$ -idempotent operator or bicomplex idempotent operator if $T^2 = T$.

Definition 4.2: $\mathbb{C}_2$ -idempotent Matrices: A Matrix $A = e_1 A^- + e_2 A^+ \in \mathbb{C}_2^{n \times n}$ is said to be $\mathbb{C}_2$ -idempotent matrix or bicomplex idempotent matrix if $A^2 = A$.

Theorem 4.3: A linear operator $T \in L_1^n \times_e L_1^n$ is a $\mathbb{C}_2$ -idempotent if and only if T_1 and T_2 are the idempotent linear operator.

Proof: Suppose $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ is a $\mathbb{C}_2$ -idempotent linear operator. We use Definition 4.1 and Theorem 2.7 throughout this proof.

$$T^2 = T$$

$$\Leftrightarrow e_1^2 T_1^2 + e_2^2 T_2^2 + 2e_1 e_2 T_1 T_2 = e_1 T_1 + e_2 T_2$$

$$\Leftrightarrow e_1 T_1^2 + e_1 T_2^2 + e_1 T_1 + e_2 T_2 \{ \because e_1 \cdot e_2 = e_2 \cdot e_1 = 0, e_1^2 = e_1 \text{ \& } e_2^2 = e_2 \}$$

$$\Leftrightarrow T_1^2 = T_1 \text{ and } T_2^2 = T_2 \quad \{ \because T^2 \text{ is L.T. \& by part (2) of Theorem 2.7} \}$$

$$\Leftrightarrow T_1 \text{ and } T_2 \text{ will be idempotent operators,}$$

as required. Thus the proof of the theorem is complete.

The following properties of C_2 -idempotent operators provide fundamental insight into their structure, composition, and algebraic significance.

Properties: Let $T = e_1 T_1 + e_2 T_2, S = e_1 S_1 + e_2 S_2$ be any two elements of $L_1^n \times_e L_1^n$. Suppose S, T are C_2 -idempotent operators. Then we have:

- (1) $I - T$ is a C_2 -idempotent operator if and only if $I - T_1, I - T_2$ are idempotent operator. Where $I = e_1 I^- + e_2 I^+$ is the identity operator.
- (2) $S \circ T$ is a C_2 -idempotent operators if and only if $S_1 \circ T_1, S_2 \circ T_2$ are idempotent operator.
- (3) $S + T$ is a C_2 -idempotent if and only if $(S_1 + T_1), (S_2 + T_2)$ are idempotent operators, provided $ST = 0, TS = 0$.

The following Theorem 4.4 true for bicomplex matrix can be verified easily.

Theorem 4.4: Let $A = e_1 A^- + e_2 A^+$ be a matrix in $C_2^{n \times n}$. Then A is a C_2 -idempotent matrix if and only if A^- and A^+ are idempotent matrix.

Proof: Suppose $A = e_1 A^- + e_2 A^+$ is a C_2 -idempotent matrix. Then by using Definition 4.2, we have

$$\begin{aligned}
 A^2 &= A \\
 \Leftrightarrow (e_1 A^- + e_2 A^+)^2 &= A_1 e_1 + A_2 e_2 \\
 \Leftrightarrow e_1^2 (A^-)^2 + e_2^2 (A^+)^2 + 2(e_1 A^-)(e_2 A^+) &= e_1 A^- + e_2 A^+ \\
 \Leftrightarrow e_1 (A^-)^2 + e_2 (A^+)^2 &= e_1 A^- + e_2 A^+ \quad \{ \because e_1 \cdot e_2 = 0, e_1^2 = e_1, e_2^2 = e_2 \} \\
 \Leftrightarrow (A^-)^2 &= A^- \text{ and } (A^+)^2 = A^+ \quad \{ \text{by Remark 2.2} \} \\
 \Leftrightarrow A^- \text{ and } A^+ &\text{ are idempotent matrices,}
 \end{aligned}$$

as required. Thus the proof of the theorem is complete.

Theorem 4.5: Let $T = e_1 T_1 + e_2 T_2 \in L_1^n \times_e L_1^n$ be a C_2 -idempotent operator and let $\mathcal{B}_1$ be the ordered basis for C_1^n such that $[T_1]_{\mathcal{B}_1} = A^-$, and $[T_2]_{\mathcal{B}_1} = A^+$ if and only if $A = e_1 A^- + e_2 A^+$ is C_2 -idempotent matrix.

Proof: Suppose $T = T_2 \in L_1^n \times_e L_1^n$ is a $\mathbb{C}_2$ -idempotent operator. Use Definitions 2.4, 4.2 and Theorem 4.3, we have

$$\Leftrightarrow T_1 \text{ and } T_2 \text{ are idempotent operator}$$

$$\Leftrightarrow \exists \text{ basis } \mathcal{B}_1 \text{ such that } [T_1]_{\mathcal{B}_1} = A^- \text{ and } [T_2]_{\mathcal{B}_1} = A^+ \text{ are idempotent matrices}$$

$$\Leftrightarrow e_1[T_1]_{\mathcal{B}_1} + e_2[T_2]_{\mathcal{B}_2} = e_1A^- + e_2A^+ \text{ is } \mathbb{C}_2\text{-idempotent matrix \{by Theorem 4.4\}.}$$

$$\Leftrightarrow A = e_1A^- + e_2A^+ \text{ is a } \mathbb{C}_2\text{-idempotent matrix,}$$

as required. Thus the proof of the theorem is complete.

Theorem 4.6: A bicomplex matrix $A = e_1A^- + e_2A^+ \in \mathbb{C}_2^{n \times n}$ is a $\mathbb{C}_2$ -idempotent matrix if and only if e_1A is a $\mathbb{C}_2$ -idempotent matrix.

Proof: Suppose $A = e_1A^- + e_2A^+$ is a $\mathbb{C}_2$ -idempotent matrix. Use Definition 4.2 and Theorem 4.4, we have

$$A^2 = A$$

$$\Leftrightarrow (A^-)^2 = A^- \text{ and } (A^+)^2 = A^+ \quad \{\text{by Theorem 4.4}\}$$

$$\text{Now, } (e_1A)^2 = e_1^2[e_1^2(A^-)^2 + e_2^2(A^+)^2 + 2A^-A^+e_1e_2]$$

$$= e_1[e_1(A^-)^2 + e_2(A^+)^2] \quad (\text{Since } e_1 \cdot e_2 = 0, e_1^2 = e_1, e_2^2 = e_2)$$

$$= e_1(e_1A^- + e_1A^+)$$

$$= e_1A$$

Hence e_1A is a $\mathbb{C}_2$ -idempotent matrix, as required. Thus the proof of the theorem is complete.

The following Corollary 4.7 is immediate consequence of Theorem 4.6.

Corollary 4.7: A bicomplex matrix $A = e_1A^- + e_2A^+ \in \mathbb{C}_2^{n \times n}$ is a $\mathbb{C}_2$ -idempotent matrix if and only if e_2A is a $\mathbb{C}_2$ -idempotent matrix.

Theorem 4.8: Let $A = e_1A^- + e_2A^+, B = e_1B^- + e_2B^+ \in \mathbb{C}_2^{n \times n}$. Then A and B are $\mathbb{C}_2$ -idempotent matrices if and only if $e_1A + e_2B$ is a $\mathbb{C}_2$ -idempotent matrix.

Proof: Suppose $A = e_1A^- + e_2A^+$ and $B = e_1B^- + e_2B^+$ are $\mathbb{C}_2$ -idempotent matrix. Use Definition 4.2, we have

$$A^2 = A \text{ and } B^2 = B.$$

$$\begin{aligned} \text{Now, } (e_1A + e_2B)^2 &= (e_1^2A^2 + e_2^2B^2 + 2(AB)(e_1e_2)) \\ &= e_1A^2 + e_2B^2 \quad \{\because e_1^2 = e_1, e_2^2 = e_2, \& e_1 \cdot e_2 = 0\} \\ &= e_1A + e_2B. \end{aligned}$$

Hence $e_1A + e_2B$ is a $\mathbb{C}_2$ -idempotent matrix, as required. Thus the proof of the theorem is complete.

Theorem 4.9: Let $A = e_1A_1 + e_2A_2$ be a $\mathbb{C}_2$ -idempotent matrix. Then $e_1(I - A)$ is also a $\mathbb{C}_2$ -idempotent matrix, where $I = e_1I^- + e_1I^+$ is identity matrix of order $n \times n$.

Proof: Suppose $A = e_1A_1 + e_2A_2$ is a $\mathbb{C}_2$ -idempotent matrix. Use Definition 4.2, we have

$$A^2 = A.$$

$$\begin{aligned} \text{Now, } [e_1(I - A)]^2 &= e_1^2(I - A)^2 \\ &= e_1^2(I^2 + A^2 - 2AI) \\ &= e_1(I + A - 2A) \quad \{\because e_1^2 = e_1\} \\ &= e_1(I - A). \end{aligned}$$

Hence $e_1(I - A)$ is a $\mathbb{C}_2$ -idempotent matrix, as required. Thus the proof of the theorem is complete.

The following Corollary 4.10 is immediate consequence of Theorem 4.9.

Corollary 4.10: Let $A = e_1A^- + e_2A^+$ be a $\mathbb{C}_2$ -idempotent matrix. Then $e_2(I - A)$ is also a $\mathbb{C}_2$ -idempotent matrix, where $I = e_1I^- + e_2I^+$ is identity matrix of order $n \times n$.

Conclusion:

In this paper, we explored the concepts of idempotent and nilpotent operators within the framework of bicomplex spaces. We also analyzed their fundamental properties and results. Additionally, we introduced the notion of idempotent matrices in bicomplex spaces and derived several important results related to their structure and properties.

The theorems establish a foundation for understanding bicomplex idempotent and nilpotent operators, highlighting their algebraic and analytical properties. These results extend matrix and operator theory to the bicomplex setting and provide a basis for further spectral theory and functional analysis research. The findings also pave the way for further research in spectral theory, functional analysis, and applications involving bicomplex structures.

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