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Application of (4, 2)-jection Operator in Exponential and Hyperbolic Functions

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Abstract:

In this paper, we use (4, 2)- jection operator which is the generalization of projection operator, projection defined in G.F. Simmons [1]. We have define e^B*, where B is a (4, 2)-jection operator on a linear space L.*

We find e^{nB}*, n being an integer. We define hyperbolic function related to a (4, 2)- jection operator B. We deduce properties of hyperbolic function such as* $cosh^2 B - sinh^2 B = I$*, etc. We prove results analogous to results in trigonometry.*

Keywords: linear space, projection operator, Hyperbolic functions, Trijection operator, (4, 2)- jection.

1. Introduction:

(4, 2)-jection and hyperbolic functions:-

Definitions:

Let B be a (4, 2)-jection operator on a linear space L.[2]
i.e. $B^4 = B^2$.

We define,

$$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^4}{4!} + \frac{B^5}{5!} + \frac{B^6}{6!} + \cdots\cdots$$

Now, $$e^B = I + B + \frac{B^2}{2!} + \frac{B^3}{3!} + \frac{B^2}{4!} + \frac{B^3}{5!} + \frac{B^2}{6!} + \cdots\cdots \quad (\text{if } B^4 = B^2)$$

$$= I + B + B^2\left(\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots\cdots\right) + B^3\left(\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\cdots\right)$$

$$= I + B + B^2\left(\frac{e^1 + e^{-1}}{2} - 1\right) + B^3\left(\frac{e^1 - e^{-1}}{2} - 1\right)$$

$$e^B = I + B + B^2(\cosh 1 - 1) + B^3(\sinh 1 - 1)$$

Therefore,

$$(e^B)^2 = [I + B + B^2(\cos h1 - 1) + B^3(\sin h1 - 1)]^2$$

$$= [I^2 + B^2 + B^4(\cos h1 - 1)^2 + B^6(\sin h1 - 1)^2 + 2.I.B$$

$$+ 2.B.B^2(\cos h1 - 1) + 2.B^2(\cos h1 - 1)B^3(\sin h1 - 1)$$

$$+ 2.I.B^3(\sin h1 - 1) + 2.I.B^2(\cos h1 - 1) + 2.B.B^3(\sin h1 - 1)]$$

$$= [I^2 + B^2 + B^2(\cos h1 - 1)^2 + B^2(\sin h1 - 1)^2 + 2.B$$

$$+ 2.B^3(\cos h1 - 1) + 2.B^3(\cos h1 - 1)(\sin h1 - 1)$$

$$+ 2.B^3(\sin h1 - 1) + 2.I.B^2(\cos h1 - 1) + 2.B.B^3(\sin h1 - 1)]$$

$$= [I^2 + B^2 + B^2(\cos^2 h1 + \sin^2 h1 - 1) + B^3(2\sin h1 \cos h1 - 2)]$$

$$= [I^2 + B^2 + B^2(\cos 2h1 - 1) + B^3(\sin 2h1 - 2)]$$

$$(e^B)^2 = [I^2 + 2B^2 + B^2(\cos 2h1 - 1) + B^3(\sin 2h1 - 2)]$$

Let us assume,

$$(e^B)^n = [I^2 + nB^2 + B^2(\cos nh1 - 1) + B^3(\sin nh1 - n)]$$

Now, $(e^B)^{n+1} = (e^B)^n \times e^B$

$$[I + B + B^2(\cos h1 - 1) + B^3(\sin h1 - 1)] \times [I^2 + nB^2 + B^2(\cos nh1 - 1) + B^3(\sin nh1 - n)]$$

$$= [I + nB + B^2(\cos nh1 - 1) + B^3(\sin nh1 - n) + B + nB^2 + B^3(\cos nh1 - 1)$$
$$+ B^4(\sin nh1 - n) + B^2(\cos h1 - 1) + nB^3(\cos h1 - 1) + B^4(\cos h1 - 1)(\cos nh1 - 1)$$
$$+ B^5(\cos h1 - 1)(\sin nh1 - n) + B^3(\sin nh1 - 1) + nB^4(\sin h1 - 1)$$
$$+ B^5(\cos nh1 - 1)(\sin h1 - n) + B^6(\sin h1 - 1)(\sin nh1 - n)]$$

$$= I + (nB + B) + B^2\{\cos nh1 - 1 + n + \sin nh1 - n + \cos h1 - 1$$
$$+ (\cos h1 - 1)(\cos nh1 - 1) + n(\sin h1 - 1) + (\sin h1 - 1)(\sin nh1 - n)\}$$
$$+ B^3\{\sin nh1 - n + \cos h1 - 1 + n(\cos h1 - 1) + (\cos h1 - 1)(\sin nh1 - n)$$
$$+ (\sin h1 - 1) + (\sin h1 - 1)(\cos nh1 - 1)\}$$

$= I + (n+1)B + B^2\{\cos h1.\cos nh1 + \sin h1.\sin nh1 - 1\} + B^3\{\cos h1.\sin nh1$
$+ \sin h1.\cos nh1 - n - 1\}$

$= I + (n+1)B + B^2\{\cos h(n+1) - 1\} + B^3\{\sin h(n+1) - (n+1)\}$

Hence by induction for any positive integer n.

$e^{nB} = [I^2 + nB^2 + B^2(\cos nh1 - 1) + B^3(\sin nh1 - n)]$

if we put $n = 0$,

$e^{0B} = [I^2 + 0.B^2 + B^2(\cos 0h1 - 1) + B^3(\sin 0h1 - 0)]$

$e^0 = [I^2 + 0 + B^2(1 - 1) + B^3(0 - 0)]$

$e^0 = I$, where 0 is zero operator.

So relation holds when $n = 0$.

So we take

$n = -n$, i.e. when n is a negative integer.

$e^{-nB} = [I + (-n)B^2 + B^2(\cos(-n)h1 - 1) + B^3(\sin(-n)h1 - (-n))]$

$e^{-nB} = [I - nB^2 + B^2(\cos nh1 - 1) - B^3(\sin nh1 - n)]$

Putting $B = I$

$e^{nI} = [I^2 + nI^2 + I^2(\cos nh1 - 1) + I^3(\sin nh1 - n)]$

$= I[I + n + \cos nh1 - 1 + \sin nh1 - n]$

$e^{nI} = I[\cos nh1 + \sin nh1]$

So we put $n = 1$

$e^I = I[\cos h1 + \sin h1]$

$e^I = Ie$ or eI

Now,

$$\cos hB = I + \frac{B^2}{2!} + \frac{B^4}{4!} + \frac{B^6}{6!} + \cdots\cdots\cdots\cdots; \text{ (if } B^4 = B^2)$$

$$= I + \frac{B^2}{2!} + \frac{B^2}{4!} + \frac{B^2}{6!} + \cdots\cdots\cdots$$

$$= I + B^2\left[\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots\cdots\cdots\right]$$

$$= I + B^2\left[\left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots\cdots\cdots\right) - 1\right]$$

$$\cos hB = I + B^2[\cos h1 - 1]$$

$$\sin hB = B + \frac{B^3}{3!} + \frac{B^5}{5!} + \frac{B^7}{7!} + \cdots\cdots\cdots\cdots; \text{ (if } B^4 = B^2)$$

$$\sin hB = B + \frac{B^3}{3!} + \frac{B^3}{5!} + \frac{B^3}{7!} + \cdots\cdots\cdots$$

$$= B + B^3\left[\frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\cdots\cdots\right]$$

$$= B + B^3\left[\left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\cdots\cdots\right) - 1\right]$$

$$\sin hB = B + B^3[\sin h1 - 1]$$

Now,

$$\sin hB + \cos hB = B + B^3[\sin h1 - 1] + I + B^2[\cos h1 - 1]$$

$$e^B = I + B + B^2[\cos h1 - 1] + B^3[\sin h1 - 1]$$

$$e^{2B} = I + 2B + B^2[\cos h2 - 1] + B^3[\sin h2 - 2]$$

$$e^{nB} = I + nB + B^2[\cos hn - 1] + B^3[\sin hn - n]$$

$$e^{nB} = \{I + B^2(\cos hn - 1)\} + \{nB + B^3(\sin hn - n)\}$$

$$e^{nB} = \sin hnB + \cos hnB$$

Theorem (1.I): $\cos h^2 nB - \sin h^2 nB = I$

Proof: $\cos hnB + \sin hnB$

$$= \{I + B^2(\cos hn - 1)\} + \{nB + B^3(\sin hn - n)\}$$

$$= e^{nB}$$

Now,

$$\cos hnB - \sin hnB$$

$$= \{I + B^2(\cos hn - 1)\} - \{nB + B^3(\sin hn - n)\}$$

$$= e^{-nB}$$

$$\cos h^2 nB - \sin h^2 nB$$

$$= (\cos hnB + \sin hnB)(\cos hnB - \sin hnB)$$

$$= e^{nB} \times e^{-nB}$$

$$= e^0$$

$$= I$$

Hence, $\cos h^2 nB - \sin h^2 nB = I$

Theorem (1.II): $\cos hnB . \sec hnB = I$

Proof: We know that

$$\sec hnB = I + B^2(\sec hn - 1)$$

$$\therefore \cos hnB . \sec hnB$$

$$= \{I + B^2(\cos hn - 1)\} . \{I + B^2(\sec hn - 1)\}$$

$$= [I + B^2(\sec hn - 1) + B^2(\cos hn - 1) + B^4(\cos hn - 1)(\sec hn - 1)] \text{ (if } B^4 = B^2)$$

$$= I + B^2[\sec hn - 1 + \cos hn - 1 + 1 + \cos hn - \sec hn + 1]$$

$$= I + B^2[0]$$

$$= I$$

Theorem (1.III): $(\cos hB + \sin hB)^n = (\cos hnB + \sin hnB)$

Proof: R.H.S

$$(\cos hnB + \sin hnB) = \{I + B^2(\cos hn - 1)\} + \{nB + B^3(\sin hn - n)\}$$
$$= [I + nB + B^2(\cos nh1 - 1) + B^3(\sin nh1 - n)]$$
$$= e^{nB} = (e^B)^n$$
$$= [I + B + B^2(\cos h1 - 1) + B^3(\sin h1 - 1)]^n$$
$$= [\{I + B^2(\cos h1 - 1)\} + \{B + B^3(\sin h1 - 1)\}]^n$$
$$= (\cos hB + \sin hB)^n$$

Theorem (1.IV): $(I + \cos h2 + \sin h2)^n = 2^n \cos h^n B(\cos hnB + \sin hnB)$

Proof: $(I + \cos h2 + \sin h2)^n = \{I + \cos h^2 B + \sin h^2 B + 2\sin hB\cos hB\}^n$

$$= (\cos^2 hB + \cos^2 hB + 2\sin hB\cos hB)^n$$
$$= \{2\cos hB(\cos hB + \sin hB)\}^n$$
$$= 2^n \cos h^n B(\cos hB + \sin hB)^n$$
$$= 2^n \cos h^n B(\cos hnB + \sin hnB)$$

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