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A brief study of some special type of sets of integers and their transformations

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Abstract:

In this paper, integers have been divided modulo 7 [16] and classes $B_1, B_2, B_3, B_4, B_5, B_6, B_7$ have been defined. We consider sums and multiplication of these sets and get a ring $(R, +, \bullet)$. We consider some linear transformation on R and show that these transformations form a group as well as a ring named K , this K is also a field [17].

Keywords: Group, Ring, Field, Modules, linear transformations

1. Definition of addition and its consequences:

Let us divided integers modulo 7. Thus, all number of the form $7K+1$, K being any integer will be denoted by $\bar{1}$ or B_1 . Number of the form $7K+2$ will be denoted by $\bar{2}$ or B_2 and so on. Thus, we get $\bar{B}_3, \bar{B}_4, \bar{B}_5$. And $\bar{B}_6$ finally number of the form $7K$ will be denoted by $\bar{7}$ or $\bar{0}$ by $\bar{B}_7$.

[61]

Then clearly,

$$B_1 + B_2 = \bar{1} + \bar{2} = \bar{3} = B_3$$

$$B_1 + B_3 = \bar{1} + \bar{3} = \bar{4} = B_4$$

$$B_1 + B_4 = \bar{1} + \bar{4} = \bar{5} = B_5$$

$$B_1 + B_5 = \bar{1} + \bar{5} = \bar{6} = B_6$$

$$B_1 + B_6 = \bar{1} + \bar{6} = \bar{7} = B_7$$

$$B_1 + B_7 = \bar{1} + \bar{7} = \bar{8} = B_8$$

$$B_2 + B_2 = \bar{2} + \bar{2} = \bar{4} = B_4 \text{ etc.}$$

Thus in general,

$$B_i + B_j = B_{i+j} \pmod{7}$$

By these calculations, we can construct the following tables

+	B_1	B_2	B_3	B_4	B_5	B_6	B_7
B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_1
B_2	B_3	B_4	B_5	B_6	B_7	B_1	B_2
B_3	B_4	B_5	B_6	B_7	B_1	B_2	B_3
B_4	B_5	B_6	B_7	B_1	B_2	B_3	B_4
B_5	B_6	B_7	B_1	B_2	B_3	B_4	B_5
B_6	B_7	B_1	B_2	B_3	B_4	B_5	B_6
B_7	B_1	B_2	B_3	B_4	B_5	B_6	B_7

In above tables, B_7 behaves like an identity element for operation ‘+’ also since

$$B_1 + B_6 = B_7 \text{ we can say that additive inverse of } B_1 \text{ is } B_6$$

$$\text{i.e. } -B_1 = B_6$$

Similarly, $-B_2 = B_5$, $-B_3 = B_4$, $-B_4 = B_3$, $-B_5 = B_2$, $-B_6 = B_1$ & $-B_7 = B_7$

as we deal with real numbers, associative law can be easily checked

$$\text{i.e.} \quad (B_1 + B_2) + B_3 = B_3 + B_3 = B_6$$

$$B_1 + (B_2 + B_3) = B_1 + B_5 = B_6$$

$$\text{Showing } (B_1 + B_2) + B_3 = B_1 + (B_2 + B_3)$$

Hence, $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$ with operator '+' is an abelian group with identity B_7 .

2. Definition of multiplication and its consequences:

Now, we define multiplication

$$B_1 B_1 = (\bar{1})(\bar{1}) = (\bar{1}) = B_1$$

$$B_1 B_2 = (\bar{1})(\bar{2}) = (\bar{2}) = B_2$$

Similarly, $B_1 B_3 = B_3$, $B_1 B_4 = B_4$, $B_1 B_5 = B_5$, $B_1 B_6 = B_6$ and $B_1 B_7 = B_7$

$$B_2 B_2 = (\bar{2})(\bar{2}) = (\bar{4}) = B_4$$

$$B_2 B_3 = (\bar{2})(\bar{3}) = (\bar{6}) = B_6$$

$$B_2 B_4 = (\bar{2})(\bar{4}) = (\bar{8}) = B_1 \text{ etc.}$$

So, in general,

$$B_i B_j = B_{ij} \pmod{7}$$

With these calculations we get the following tables.

•	B_1	B_2	B_3	B_4	B_5	B_6	B_7
B_1	B_1	B_2	B_3	B_4	B_5	B_6	B_7
B_2	B_2	B_4	B_6	B_1	B_3	B_5	B_7
B_3	B_3	B_6	B_2	B_5	B_1	B_4	B_7

B_4	B_4	B_1	B_5	B_2	B_6	B_3	B_7
B_5	B_5	B_3	B_1	B_6	B_4	B_2	B_7
B_6	B_6	B_5	B_4	B_3	B_2	B_1	B_7
B_7	B_7	B_7	B_7	B_7	B_7	B_7	B_7

Hear with operation ' $\bullet$ ' B_1 behaves an identity element

Since,

$$B_1 B_1 = B_1$$

$$B_2 B_4 = B_1$$

$$B_3 B_5 = B_1$$

$$B_4 B_2 = B_1$$

$$B_5 B_3 = B_1 \text{ etc.}$$

We can write

$$B_1^{-1} = B_1$$

$$B_2^{-1} = B_4$$

$$B_3^{-1} = B_5$$

$$B_4^{-1} = B_2 \text{ etc.}$$

We can check associative laws

$$\text{e.g. } (B_1 B_2) B_3 = B_2 B_3 = B_6$$

$$B_1 (B_2 B_3) = B_1 B_6 = B_6$$

$$(B_2 B_3) B_4 = B_6 B_4 = B_3$$

$$B_2 (B_3 B_4) = B_2 B_5 = B_3 \text{ etc.}$$

Hence $\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}$ together with ' $\bullet$ ' forms an abelian group of with $\{B_1, B_6\}$ is a subgroup.

Now, we check distributive laws e.g.

$$B_1(B_2 B_3) = B_1 B_5 = B_5, \quad B_1 B_2 + B_1 B_3 = B_2 + B_3 = B_5$$

$$B_2(B_3 B_4) = B_2 B_7 = B_7, \quad B_2 B_3 + B_2 B_4 = B_6 + B_1 = B_7$$

Hence considering previous we see that $(\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}, +, \bullet) = R$ (say) is an abelian ring with multiplication identity $B_1 (\neq B_7)$ and B_7 being identity for '+' also R has no zero divisors, so it is an integral domain. We know that every finite integral domain is a field. Hence it is a field also.

Now we consider powers of the elements of R .

$$B_1^2 = B_1, \quad B_2^2 = B_4, \quad B_3^2 = B_2, \quad B_4^2 = B_2,$$

$$B_5^2 = B_4, \quad B_6^2 = B_1, \quad B_7^2 = B_7$$

$$B_1^3 = B_1^2 \cdot B_1 = B_1 \cdot B_1 = B_1, \quad B_2^3 = B_2^2 \cdot B_2 = B_4 \cdot B_2 = B_1$$

$$B_3^3 = B_3^2 \cdot B_3 = B_2 \cdot B_3 = B_6, \quad B_4^3 = B_4^2 \cdot B_4 = B_2 \cdot B_4 = B_1$$

$$B_5^3 = B_5^2 \cdot B_5 = B_4 \cdot B_5 = B_6, \quad B_6^3 = B_6^2 \cdot B_6 = B_1 \cdot B_6 = B_6$$

$$B_7^3 = B_7^2 \cdot B_7 = B_7 \cdot B_7 = B_7$$

$$B_1^4 = B_1^3 \cdot B_1 = B_1 \cdot B_1 = B_1, \quad B_2^4 = B_2^3 \cdot B_2 = B_1 \cdot B_2 = B_2, \quad B_3^4 = B_3^3 \cdot B_3 = B_6$$

$$B_3 = B_4, \quad B_4^4 = B_4^3 \cdot B_4 = B_1 \cdot B_4 = B_4, \quad B_5^4 = B_5^3 \cdot B_5 = B_6 \cdot B_5 = B_2, \quad B_6^4 = B_6^3 \cdot B_6$$

$$= B_6 \cdot B_6 = B_1, \quad B_7^4 = B_7^3 \cdot B_7 = B_7 \cdot B_7 = B_7$$

$$B_1^5 = B_1^4 \cdot B_1 = B_1 \cdot B_1 = B_1, \quad B_2^5 = B_2^4 \cdot B_2 = B_2 \cdot B_2 = B_4$$

$$B_3^5 = B_3^4 \cdot B_3 = B_4 \cdot B_3 = B_5, \quad B_4^5 = B_4^4 \cdot B_4 = B_4 \cdot B_4 = B_2$$

$$B_5^5 = B_5^4 \cdot B_5 = B_2 \cdot B_5 = B_3, \quad B_6^5 = B_6^4 \cdot B_6 = B_1 \cdot B_6 = B_6$$

$$B_7^5 = B_7^4 \cdot B_7 = B_7 \cdot B_7 = B_7$$

$$B_1^6 = B_1^5 \cdot B_1 = B_1 \cdot B_1 = B_1, \quad B_2^6 = B_2^5 \cdot B_2 = B_4 \cdot B_2 = B_1$$

$$B_3^6 = B_3^5 \cdot B_3 = B_5 \cdot B_3 = B_1, \quad B_4^6 = B_4^5 \cdot B_4 = B_2 \cdot B_1 = B_2$$

$$B_5^6 = B_5^5 \cdot B_5 = B_3 \cdot B_5 = B_1, \quad B_6^6 = B_6^5 \cdot B_6 = B_6 \cdot B_6 = B_1$$

$$\begin{aligned}
B_7^6 &= B_7^5 \cdot B_7 = B_7 \cdot B_7 = B_7 \\
B_1^7 &= B_1^6 \cdot B_1 = B_1 \cdot B_1 = B_1, & B_2^7 &= B_2^6 \cdot B_2 = B_1 \cdot B_2 = B_2 \\
B_3^7 &= B_3^6 \cdot B_3 = B_1 \cdot B_3 = B_3, & B_4^7 &= B_4^6 \cdot B_4 = B_2 \cdot B_4 = B_1 \\
B_5^7 &= B_5^6 \cdot B_5 = B_1 \cdot B_5 = B_5, & B_6^7 &= B_6^6 \cdot B_6 = B_1 \cdot B_6 = B_6 \\
B_7^7 &= B_7^6 \cdot B_7 = B_7 \cdot B_7 = B_7
\end{aligned}$$

So, in general,

$$B_i^7 = B_i (i = 1, 2, 3, \dots, 7)$$

Also $B_i^8 = B_i^7 \cdot B_i = B_i$ $B_i^9 = B_i^2$ etc.

So, we conclude that $B_i^n = B_i^m$ where $n = m \pmod{6}$

We get tables for powers of B_i

Index	2	3	4	5	6	7
B_1	B_1	B_1	B_1	B_1	B_1	B_1
B_2	B_4	B_1	B_2	B_4	B_1	B_2
B_3	B_2	B_6	B_4	B_5	B_1	B_3
B_4	B_2	B_1	B_4	B_2	B_2	B_1
B_5	B_4	B_6	B_2	B_3	B_1	B_5
B_6	B_1	B_6	B_1	B_6	B_1	B_6
B_7	B_7	B_7	B_7	B_7	B_7	B_7

Next, we consider integral multiples of B_i 's

$$2B_1 = B_1 + B_1 = B_2$$

$$2B_2 = B_2 + B_2 = B_4$$

$$2B_3 = B_3 + B_3 = B_6$$

$$2B_4 = B_4 + B_4 = B_8 \equiv B_1$$

$$2B_5 = B_5 + B_5 = B_{10} \equiv B_3$$

$$2B_6 = B_6 + B_6 = B_{12} \equiv B_5$$

$$2B_7 = B_7 + B_7 = B_{14} \equiv B_7$$

$$3B_1 = 2B_1 + B_1 = B_2 + B_1 = B_3$$

$$3B_2 = 2B_2 + B_2 = B_4 + B_2 = B_6$$

$$3B_3 = 2B_3 + B_3 = B_6 + B_3 = B_9 \equiv B_2 \text{ etc.}$$

So, in general,

$$MB_i = B_j \text{ where } j = M_i \pmod{7}$$

We can make a table regarding integral multiples as following

•	B_1	B_2	B_3	B_4	B_5	B_6	B_7
2	B_2	B_4	B_6	B_1	B_3	B_5	B_7
3	B_3	B_6	B_2	B_5	B_1	B_4	B_7
4	B_4	B_1	B_5	B_2	B_6	B_3	B_7
5	B_5	B_3	B_1	B_6	B_4	B_2	B_7
6	B_6	B_5	B_4	B_3	B_2	B_1	B_7
7	B_7	B_7	B_7	B_7	B_7	B_7	B_7

We show that

$$(i + j)B_k = iB_k + jB_k$$

e.g. $(1 + 3)B_1 = 4B_1 = B_4$

$$B_1 + 3B_1 = B_1 + B_3 = B_4 \text{ etc.}$$

We have property such that

$$a(B_i + B_j) = aB_i + aB_j$$

e.g. $2(B_2 + B_4) = 2B_6 = B_5$

$$2B_2 + 2B_4 = B_4 + B_1 = B_5 \text{ etc.}$$

We have property such that

$$(ab)B_i = a(bB_i)$$

e.g. $(2 \times 3)B_1 = 6B_1 = B_6$

$$2(3B_1) = 2B_3 = B_6 \text{ etc.}$$

Also, 1. $B_i = B_i$

So, the system $(\{B_1, B_2, B_3, B_4, B_5, B_6, B_7\}, +, \bullet)$ forms a system analogous to a vector space over the set of integers.

Let us call it V .

3. Some linear transformations on R :

Define $S : R \rightarrow R$ by

$$S(B_i) = B_i B_2 \text{ (i.e. multiplying by 2, 9,....., etc)}$$

Then,

$$S(B_1) = B_1 B_2 = B_2$$

$$S(B_2) = B_2 B_2 = B_4$$

$$S(B_3) = B_3 B_2 = B_6$$

$$S(B_4) = B_4 B_2 = B_1$$

$$S(B_5) = B_5 B_2 = B_3$$

$$S(B_6) = B_6 B_2 = B_5$$

$$S(B_7) = B_7 B_2 = B_7$$

We show that S is linear, i.e.

$$S(B_i + B_j) = S(B_i) + S(B_j) \text{ and } S(\alpha B_i) = \alpha S(B_i)$$

Now, $S(B_i + B_j) = (B_i + B_j)B_2 = B_i B_2 + B_j B_2 = S(B_i) + S(B_j)$

And $S(\alpha B_i) = \alpha(B_i) B_2 = \alpha(B_i B_2) = \alpha S(B_i)$

Where α varies from 1 to 5

Also $S^2(B_i) = S(S(B_i)) = S(B_i B_2) = (B_i B_2) B_2 = B_i (B_2 B_2) = B_i (B_4) = B_i B_4$

$S^3(B_i) = S(S^2(B_i)) = S(B_i B_4) = (B_i B_4) B_2 = B_i (B_4 B_2) = B_i (B_1) = B_i B_1 = B_i$

$S^3 = I$, identity transformation on R .

$S^4(B_i) = S(S^3(B_i)) = S(B_i B_1) = (B_i B_1) B_2 = B_i (B_1 B_2) = B_i (B_2) = B_i B_2$

$S^4 = S$, i.e. S is tetrajection

$S^5(B_i) = S(S^4(B_i)) = S(B_i B_2) = (B_i B_2) B_2 = B_i (B_2 B_2) = B_i (B_4) = B_i B_4$

$S^6(B_i) = S(S^5(B_i)) = S(B_i B_4) = (B_i B_4) B_2 = B_i (B_4 B_2) = B_i (B_1) = B_i$

So, we conclude that

$S^6 = S^3 = I$, Identity transformation on R .

Therefore $S^7 = S$

Next, we define $T : R \rightarrow R$ by

$T(B_i) = B_i B_3$ (i.e. multiplying by 3, 10,....., etc)

Then,

$T(B_1) = B_1 B_3 = B_3$

$T(B_2) = B_2 B_3 = B_6$

$T(B_3) = B_3 B_3 = B_2$

$T(B_4) = B_4 B_3 = B_5$

$T(B_5) = B_5 B_3 = B_1$

$T(B_6) = B_6 B_3 = B_4$

$T(B_7) = B_7 B_3 = B_7$

As in case of T , we can easily see that T is linear also

$$\begin{aligned} T^2(B_i) &= T(T(B_i)) = T(B_i B_3) = (B_i B_3) B_3 = B_i (B_3 B_3) = B_i (B_2) = B_i B_2 \\ T^3(B_i) &= T(T^2(B_i)) = T(B_i B_2) = (B_i B_2) B_3 = B_i (B_2 B_3) = B_i (B_6) = B_i B_6 \\ T^4(B_i) &= T(T^3(B_i)) = T(B_i B_6) = (B_i B_6) B_3 = B_i (B_6 B_3) = B_i (B_4) = B_i B_4 \\ T^5(B_i) &= T(T^4(B_i)) = T(B_i B_4) = (B_i B_4) B_3 = B_i (B_4 B_3) = B_i (B_5) = B_i B_5 \\ T^6(B_i) &= T(T^5(B_i)) = T(B_i B_5) = (B_i B_5) B_3 = B_i (B_5 B_3) = B_i (B_1) = B_i \end{aligned}$$

So, we conclude that

$$T^6 = I, \text{ Identity transformation on } R.$$

Therefore $T^7 = T$

Next, we define $U : R \rightarrow R$ by

$$U(B_i) = B_i B_4 \text{ (i.e. multiplying by 4, 11, ..., etc)}$$

Then,

$$\begin{aligned} U(B_1) &= B_1 B_4 = B_4 \\ U(B_2) &= B_2 B_4 = B_1 \\ U(B_3) &= B_3 B_4 = B_5 \\ U(B_4) &= B_4 B_4 = B_2 \\ U(B_5) &= B_5 B_4 = B_6 \\ U(B_6) &= B_6 B_4 = B_3 \\ U(B_7) &= B_7 B_4 = B_7 \end{aligned}$$

As in case of U , we can easily see that U is linear also

$$\begin{aligned} U^2(B_i) &= U(U(B_i)) = U(B_i B_4) = (B_i B_4) B_4 = B_i (B_4 B_4) = B_i (B_2) = B_i B_2 \\ U^3(B_i) &= U(U^2(B_i)) = U(B_i B_2) = (B_i B_2) B_4 = B_i (B_2 B_4) = B_i (B_8) = B_i B_1 = B_i \\ U^3 &= I, \text{ Identity transformation on } R. \end{aligned}$$

$$U^4(B_i) = U(U^3(B_i)) = U(B_i B_1) = (B_i B_1) B_4 = B_i (B_1 B_4) = B_i (B_4) = B_i B_4$$

$$U^4 = U, \text{ i.e. } S \text{ is tetrajection.}$$

$$U^5(B_i) = U(U^4(B_i)) = U(B_i B_4) = (B_i B_4) B_4 = B_i (B_4 B_4) = B_i (B_2) = B_i B_2$$

$$U^6(B_i) = U(U^5(B_i)) = U(B_i B_2) = (B_i B_2) B_4 = B_i (B_2 B_4) = B_i (B_1) = B_i B_1 = B_i$$

$$U^6 = U^3 = I, \text{ Identity transformation on } R.$$

$$U^7(B_i) = U(U^6(B_i)) = U(B_i B_1) = (B_i B_1) B_4 = B_i (B_1 B_4) = B_i (B_4) = B_i B_4$$

$$\text{Hence } U^4 = U^4 = U$$

We define $V : R \rightarrow R$ by

$$V(B_i) = B_i B_5 \text{ (i.e. multiplying by 5, 12, ..., etc)}$$

Then,

$$V(B_1) = B_1 B_5 = B_5$$

$$V(B_2) = B_2 B_5 = B_3$$

$$V(B_3) = B_3 B_5 = B_1$$

$$V(B_4) = B_4 B_5 = B_6$$

$$V(B_5) = B_5 B_5 = B_4$$

$$V(B_6) = B_6 B_5 = B_2$$

$$V(B_7) = B_7 B_5 = B_7$$

As in case of V , we can easily see that V is linear also

$$V^2(B_i) = V(V(B_i)) = V(B_i B_5) = (B_i B_5) B_5 = B_i (B_5 B_5) = B_i (B_4) = B_i B_4$$

$$V^3(B_i) = V(V^2(B_i)) = V(B_i B_4) = (B_i B_4) B_5 = B_i (B_4 B_5) = B_i (B_6) = B_i B_6$$

$$V^4(B_i) = V(V^3(B_i)) = V(B_i B_6) = (B_i B_6) B_5 = B_i (B_6 B_5) = B_i (B_2) = B_i B_2$$

$$V^5(B_i) = V(V^4(B_i)) = V(B_i B_2) = (B_i B_2) B_5 = B_i (B_2 B_5) = B_i (B_3) = B_i B_3$$

$$V^6(B_i) = V(V^5(B_i)) = V(B_i B_3) = (B_i B_3) B_5 = B_i (B_3 B_5) = B_i (B_1) = B_i B_1 = B_i$$

$$V^7(B_i) = V(V^6(B_i)) = V(B_i B_1) = (B_i B_1) B_5 = B_i (B_1 B_5) = B_i (B_5) = B_i B_5$$

Hence $V^6 = I$, Identity transformation on R and $V^7 = V$

We define $W : R \rightarrow R$ by

$$W(B_i) = B_i B_6 \text{ (i.e. multiplying by 6, 13, ..., etc)}$$

Then,

$$W(B_1) = B_1 B_6 = B_6$$

$$W(B_2) = B_2 B_6 = B_5$$

$$W(B_3) = B_3 B_6 = B_4$$

$$W(B_4) = B_4 B_6 = B_3$$

$$W(B_5) = B_5 B_6 = B_2$$

$$W(B_6) = B_6 B_6 = B_1$$

$$W(B_7) = B_7 B_6 = B_7$$

As in case of W , we can easily see that W is linear also

$$W^2(B_i) = W(W(B_i)) = W(B_i B_6) = (B_i B_6) B_6 = B_i (B_6 B_6) = B_i (B_1) = B_i B_1 = B_i$$

$$W^3(B_i) = W(W^2(B_i)) = W(B_i B_1) = (B_i B_1) B_6 = B_i (B_1 B_6) = B_i (B_6) = B_i B_6$$

$$W^3 = W, \text{ i.e. } W \text{ is trijection}$$

$$W^4(B_i) = W(W^3(B_i)) = W(B_i B_6) = (B_i B_6) B_6 = B_i (B_6 B_6) = B_i (B_1) = B_i B_1$$

Hence $W^4 = W^2 = I$, Identity transformation on R

$$W^5(B_i) = W(W^4(B_i)) = W(B_i B_1) = (B_i B_1) B_6 = B_i (B_1 B_6) = B_i (B_6) = B_i B_6$$

Hence $W^5 = W$, i.e. W is pantajection

$$W^6(B_i) = W(W^5(B_i)) = W(B_i B_6) = (B_i B_6) B_6 = B_i (B_6 B_6) = B_i (B_1) = B_i B_1 = B_i$$

Hence $W^6 = W^4 = W^2 = I$

$$W^7(B_i) = W(W^6(B_i)) = W(B_i B_1) = (B_i B_1) B_6 = B_i (B_1 B_6) = B_i (B_6) = B_i B_6$$

Hence $W^7 = W^3 = W$

Since $S^2(B_i) = B_i B_4$

Hence $S^2 = U$.

We show that S and U commute since

$$(SU)B_i = S(U(B_i)) = S(B_i B_4) = (B_i B_4)B_2 = B_i(B_4 B_2) = B_i B_1 = B_i$$

$$\text{And } (US)B_i = U(S(B_i)) = U(B_i B_2) = (B_i B_2)B_4 = B_i(B_2 B_4) = B_i B_1 = B_i$$

$$\text{So, } SU = US = I$$

We show that S and T commute since

$$(ST)B_i = S(T(B_i)) = S(B_i B_3) = (B_i B_3)B_2 = B_i(B_3 B_2) = B_i B_6 = W$$

$$\text{And } (TS)B_i = T(S(B_i)) = T(B_i B_2) = (B_i B_2)B_3 = B_i(B_2 B_3) = B_i B_6 = W$$

$$\text{Hence } ST = TS = W$$

We show that S and V commute since

$$(SV)B_i = S(V(B_i)) = S(B_i B_5) = (B_i B_5)B_2 = B_i(B_5 B_2) = B_i B_3 = T$$

$$\text{And } (VS)B_i = V(S(B_i)) = V(B_i B_2) = (B_i B_2)B_5 = B_i(B_2 B_5) = B_i B_3 = T$$

$$\text{Hence } SV = VS = T$$

We show that S and W commute since

$$(SW)B_i = S(W(B_i)) = S(B_i B_6) = (B_i B_6)B_2 = B_i(B_6 B_2) = B_i B_5 = V$$

$$\text{And } (WS)B_i = W(S(B_i)) = W(B_i B_2) = (B_i B_2)B_6 = B_i(B_2 B_6) = B_i B_5 = V$$

$$\text{Hence } SW = WS = V$$

We show that T and U commute since

$$(TU)B_i = T(U(B_i)) = T(B_i B_4) = (B_i B_4)B_3 = B_i(B_4 B_3) = B_i B_5 = V$$

$$\text{And } (UT)B_i = U(T(B_i)) = U(B_i B_3) = (B_i B_3)B_4 = B_i(B_3 B_4) = B_i B_5 = V$$

$$\text{Hence } TU = UT = VS$$

We show that T and V commute since

$$(TV)B_i = T(V(B_i)) = T(B_i B_5) = (B_i B_5)B_3 = B_i(B_5 B_3) = B_i B_1 = B_i = I$$

$$\text{And } (VT)B_i = V(T(B_i)) = V(B_i B_3) = (B_i B_3)B_5 = B_i(B_3 B_5) = B_i B_1 = B_i = I$$

Hence $TV = VT = I$

We show that T and W commute since

$$(TW)B_i = T(W(B_i)) = T(B_i B_6) = (B_i B_6)B_3 = B_i(B_6 B_3) = B_i B_4 = U$$

$$\text{And } (WT)B_i = W(T(B_i)) = W(B_i B_3) = (B_i B_3)B_6 = B_i(B_3 B_6) = B_i B_4 = U$$

Hence $TW = WT = U$

We show that U and V commute since

$$(UV)B_i = U(V(B_i)) = U(B_i B_5) = (B_i B_5)B_4 = B_i(B_5 B_4) = B_i B_6 = W$$

$$\text{And } (VU)B_i = V(U(B_i)) = V(B_i B_4) = (B_i B_4)B_5 = B_i(B_4 B_5) = B_i B_6 = W$$

Hence $UV = VU = W$.

We show that U and W commute since

$$(UW)B_i = U(W(B_i)) = U(B_i B_6) = (B_i B_6)B_4 = B_i(B_6 B_4) = B_i B_3 = T$$

$$\text{And } (WU)B_i = W(U(B_i)) = W(B_i B_4) = (B_i B_4)B_6 = B_i(B_4 B_6) = B_i B_3 = T$$

Hence $UW = WU = T$

We show that V and W commute since

$$(VW)B_i = V(W(B_i)) = V(B_i B_6) = (B_i B_6)B_5 = B_i(B_6 B_5) = B_i B_2 = S$$

$$\text{And } (WV)B_i = W(V(B_i)) = W(B_i B_5) = (B_i B_5)B_6 = B_i(B_5 B_6) = B_i B_2 = S$$

Hence $VW = WV = S$

Also, since $S^2(B_i) = U = V^2(B_i)$

i.e. $S^2 = V^2 = U$

And $T^2(B_i) = U^2(B_i) = S$

i.e. $T^2 = U^2 = S$

or $T^2(B_i) = W^2(B_i) = I$

$$I = W$$

So, combining above results.

$$ST = TS = UV = VU = W$$

$$SV = VS = UW = WU = T$$

$$SW = WS = TU = UT = V$$

$$T^2 = U^2 = VW = WV = S$$

$$S^2 = V^2 = TW = WT = U$$

We can verify associative laws e.g.

$$S(TU) = SV = T$$

or $(ST)U = WU = T$ etc

Hence, we make a table with I, S, T, U, V, W , and multiplication as binary operation

$\bullet$	I	S	T	U	V	W
I	I	S	T	U	V	W
S	S	U	W	I	T	V
T	T	W	S	V	I	U
U	U	I	V	S	W	T
V	V	T	I	W	U	S
W	W	V	U	T	S	I

Thus set (I, S, T, U, V, W) form an abelian group with identity I where,

$$I^{-1}=I, \quad S^{-1}=U, \quad U^{-1}=S, \quad T^{-1}=V, \quad V^{-1}=T, \quad W^{-1}=W$$

Associative law for multiplication has been also checked.

Now, we consider sum of operators

$$(S+S)B_i = (S)B_i + (S)B_i = B_iB_2 + B_iB_2 = B_i(B_2+B_2) = B_iB_4 = U(B_i)$$

Hence $S+S=U$

$$(T+T)B_i = T(B_i) + T(B_i) = B_iB_3 + B_iB_3 = B_i(B_3+B_3) = B_iB_6 = W(B_i)$$

Hence $T+T=W$

$$(U+U)B_i = U(B_i) + U(B_i) = B_iB_4 + B_iB_4 = B_i(B_4+B_4) = B_iB_8 = B_iB_1 = I$$

Hence $U+U=I$

$$(V+V)B_i = V(B_i) + V(B_i) = B_iB_5 + B_iB_5 = B_i(B_5+B_5) = B_iB_3 = T$$

Hence $V+V=T$

$$(W+W)B_i = W(B_i) + W(B_i) = B_iB_6 + B_iB_6 = B_i(B_6+B_6) = B_iB_5 = V$$

Hence $W+W=V$

$$(I+I)B_i = I(B_i) + I(B_i) = B_i + B_i$$

$$\text{So, } (I+I)B_1 = I(B_1) + I(B_1) = B_1 + B_1 = B_2 = S(B_1)$$

$$(I+I)B_2 = I(B_2) + I(B_2) = B_2 + B_2 = B_4 = S(B_2)$$

$$(I+I)B_3 = I(B_3) + I(B_3) = B_3 + B_3 = B_6 = S(B_3)$$

$$(I+I)B_4 = I(B_4) + I(B_4) = B_4 + B_4 = B_8 = S(B_4)$$

$$(I+I)B_5 = I(B_5) + I(B_5) = B_5 + B_5 = B_3 = S(B_5)$$

$$(I+I)B_6 = I(B_6) + I(B_6) = B_6 + B_6 = B_5 = S(B_6)$$

$$(I+I)B_7 = I(B_7) + I(B_7) = B_7 + B_7 = B_7 = S(B_7)$$

Hence, $I+I=S$

$$(S + V)B_i = S(B_i) + V(B_i) = B_iB_2 + B_iB_5 = B_i(B_2 + B_5) = B_iB_7 = B_7 = 0(B_i)$$

where 0 is zero operators such that $0(B_i) = B_7$

Hence, $S + V = 0$

$$(S + W)B_i = S(B_i) + W(B_i) = B_iB_2 + B_iB_6 = B_i(B_2 + B_6) = B_iB_8 = B_iB_1 = B_i = I$$

Hence, $S + W = I$

$$(T + W)B_i = T(B_i) + W(B_i) = B_iB_3 + B_iB_6 = B_i(B_3 + B_6) = B_iB_9 = B_iB_2 = S(B_i) = S$$

Hence, $T + W = S$

Also,
$$I + S = (S + W) + S = S + (W + S) = S + (S + W) = (S + S) + W = U + W = T$$

Similarly,
$$I + T = (S + W) + T = S + (W + T) = S + S = U$$

$$I + U = (S + W) + U = S + (W + U) = S + T = V$$

$$I + V = (S + W) + V = S + (W + V) = S + U = W$$

$$I + W = (S + W) + W = S + (W + W) = S + V = O$$

Hence when we consider $\{0, I, S, T, U, V, W\}$ with '+' we obtain the following table.

+	0	I	S	T	U	V	W
0	0	I	S	T	U	V	W
I	I	S	T	U	V	W	0
S	S	T	U	V	W	0	I
T	T	U	V	W	0	I	S
U	U	V	W	0	I	S	T
V	V	W	0	I	S	T	U
W	W	0	I	S	T	U	V

Thus, it is an abelian group with 0 as identity element.

Also, $-I = W$, $-S = V$, $-T = U$, $-U = T$, $-V = S$, $-W = I$

Distributive laws with $+$ and $\bullet$ can be verified

e.g. $T(S + T) = TV = I$

$TS + TT = W + S = I$ etc.

Also, since $(\{I, S, T, U, V, W\}, \bullet)$ forms an abelian group does not have zero divisors and distributive laws hold, hence the system, $K = (\{0, I, S, T, U, V, W\}, +, \bullet)$ forms an abelian ring with identity I , zero 0 and no zero divisors. Thus, it forms an integral domain.

Since every finite integral domain is a field, it is also a field.

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