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Diffraction of Plane P-Waves by a Fluid-Filled Circular Cylinder in an Infinite Elastic Medium

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Abstract:

The problem of diffraction of time-harmonic plane compressional (P) waves by a circular cylindrical inclusion filled with an inviscid compressible fluid and embedded in an infinite, homogeneous, isotropic elastic medium is investigated analytically using the wave function expansion method pioneered by Sezawa (1927) and later developed by Pao and Mow (1973). The incident P-wave potential and scattered wave potentials are expressed in terms of Bessel and Hankel functions. The continuity of radial displacement and traction at the fluid-solid interface is enforced to obtain the scattering coefficients. The scattering cross-section is analyzed numerically. It is observed that fluid bulk modulus significantly influences the low-frequency scattering behavior.

Keywords: P-wave diffraction, fluid-filled cylinder, elastic medium, wave function expansion, scattering coefficients, Bessel's function

Mathematics Subject Classification (2020): 74J20, 74B05, 74F10, 74J10.

1. Introduction:

The study of diffraction of elastic waves by cylindrical obstacles is a fundamental problem in elastodynamics with applications in geophysics, non-destructive testing, and acoustic engineering.

The foundations of elastodynamics were laid by Navier (1821) and Cauchy (1823), who formulated the general equations of motion for an isotropic elastic solid. Poisson (1829) first predicted the existence of two types of elastic waves: dilatational (longitudinal, P) and distortional (transverse, S). The mathematical theory was further advanced by Stokes (1845) with his solution for wave propagation in an infinite medium.

The specific problem of wave propagation in a cylindrical structure was first solved by Pochhammer (1876) and Chree (1889). Later, Sezawa (1927) studied scattering of elastic waves by a cylindrical cavity. A major breakthrough was given by Biot (1956), who studied the interaction of elastic waves with fluid-saturated porous media, which provides the basis for fluid-solid interaction.

In the present work, the scattering problem is revisited for a circular cylindrical inclusion filled with an inviscid compressible fluid. Emphasis is placed on deriving the exact scattering coefficients and analyzing the low-frequency behaviour of the scattering cross-section as a function of the fluid bulk modulus.

The diffraction by a fluid cylinder was systematically treated by Pao and Mow (1960, 1973) in their classical book "Diffraction of Elastic Waves and Dynamic Stress Concentrations". They used the separation of variables and expansion in cylindrical wave function. Following their approach, White (1983) and Schoenberg (1983, 1984) extended the theory to acoustic logging in boreholes.

The time domain substructure method proposed by Rahman and Bhatta (2022) for water-cylinder interaction provides a modern numerical approach.

Despite extensive work, the transient and impulsive loading case, especially for fluid-filled inclusion, still requires detailed analysis for dynamic stress concentration, which is the main motivation for the present work.

2. Definitions and Preliminaries:

Definition 2.1 (Plane P-Wave):

A plane compressional wave is an irrotational wave where particle displacement $\boldsymbol{u}$ is given by

$$\boldsymbol{u} = \nabla\phi$$

and scalar potential ϕ satisfies the Helmholtz equation

$$\nabla^2\phi + k_p^2\phi = 0,$$

where $\boldsymbol{u}$ is the particle displacement vector, ϕ is the scalar potential, ∇ is the gradient operator, ∇^2 is the Laplacian operator, $k_p = \omega / c_p$ is the P-wave number, ω is the angular frequency, and c_p is the P-wave velocity.

Definition 2.2 (Fluid-filled Circular Cylinder):

An infinite circular cylinder of radius a, filled with an inviscid and compressible fluid characterized by density ρ_f and bulk modulus k_f, with acoustic wave velocity $c_f = \sqrt{k_f / \rho_f}$. The fluid cannot sustain shear stress. Here ρ_f is density of fluid, k_f is bulk modulus of fluid.

Definition 2.3 (Infinite Elastic Medium):

A homogeneous, isotropic, linearly elastic medium with Lamé constants λ and μ, and density ρ, governed by Navier's equation of motion:

$$(\lambda + \mu)\,\nabla(\nabla . \boldsymbol{u}) + \mu\nabla^2\,\boldsymbol{u} = \rho\,\frac{\partial^2 \boldsymbol{u}}{\partial t^2}$$

i.e. $$(\lambda + 2\mu)\,\nabla(\nabla . \boldsymbol{u}) - \mu\nabla x(\lambda \times \mu) = \rho\,\ddot{\boldsymbol{u}}$$

Definition 2.4 (Scattering Coefficients):

Complex coefficients A_n and B_n appearing in the cylindrical wave function expansion of the scattered P- and SV-waves in the elastic medium, which are determined from the boundary conditions at $r = a$ and determine the amplitude of the nth order cylinder harmonic in the elastic medium.

Where,

A_n = scattering coefficient for the nth order scattered P-wave,
B_n = scattering coefficient for the nth order scattered SV-wave,
n = order of cylindrical harmonics ($n = 0, 1, 2, ...$)
a = radius of cylindrical inclusion,
$r = a$ = boundary condition location.

3. Mathematical Formulation:

Let the infinite elastic medium occupies $r > a$ and the fluid occupies $r < a$ in cylindrical coordinates (r, θ, z). We consider plane strain, $\frac{\partial}{\partial z} = 0$.

3.1 Governing Equations:

As per Helmholtz (1858) decomposition:

$$\boldsymbol{u} = \nabla\phi + \nabla x(\psi\, e_z)$$

where potentials satisfy:

$$\nabla^2\phi + k_p^2\phi = 0,\ k_p = \omega/c_p,\ \ c_p = \sqrt{(\lambda + 2\mu)/\rho}$$

$$\nabla^2\psi + k_s^2\psi = 0,\ k_s = \omega/c_s,\ \ c_s = \sqrt{\mu/\rho}$$

For the fluid, acoustic pressure

$p = -\rho_f \omega^2 \phi_f$ and ϕ_f satisfies:

$$\nabla^2\phi_f + k_f^2\phi_f = 0,\ k_f = \omega/c_f$$

3.2 Wave Function Expansion (Pao & Mow, 1973):

Incident P-wave propagating along the x-axis:

$$\phi_{inc} = \phi_0 \sum_{n=0}^{\infty} \varepsilon_n \, i^n J_n (k_p r) \cos n\theta$$

where $\varepsilon_0 = 1$, $\varepsilon_n = 2$ for $n \geq 1$, and J_n is the Bessel function of the first kind.

Scattered Fields:

$$\phi_{sc} = \sum_{n=0}^{\infty} A_n H_n^{(1)}(k_p r) \cos n\theta$$

$$\Psi_{sc} = \sum_{n=0}^{\infty} B_n H_n^{(1)}(k_s r) \sin n\theta$$

where $H_n^{(1)}$ is the Hankel function of the first kind (outgoing wave).

Inside Fluid:

$$\phi_f = \sum_{n=0}^{\infty} C_n J_n (k_f r) \cos n\theta$$

3.3 Boundary Conditions at $r = a$:

At the fluid-solid interface $r = a$, the following conditions hold:

1. Continuity of radial displacement: $u_r = u_r^f$.

2. Continuity of radial stress: $\sigma_{rr} = -p$, where p is acoustic pressure in the fluid.

3. Vanishing of shear stress: $\sigma_{r\theta} = 0$, since the inside liquid is inviscid (Biot, 1956).

Using these conditions, we obtain a 3×3 linear system for each n to find A_n, B_n, and C_n.

4. Theorems, Lemma and Corollary:

Lemma 1 (Orthogonality of trigonometric functions):

$$\text{For } m, n \geq 0, \quad \int_0^{2\pi} \cos m\theta \, . \cos n\theta \, d\theta = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \\ 2\pi, & m = n = 0 \end{cases}$$

$$\int_0^{2\pi} \sin m\theta \,.\, \sin n\theta \, d\theta = \begin{cases} 0, & m \neq n \\ \pi, & m = n \neq 0 \end{cases}$$

$$\& \qquad \int_0^{2\pi} \sin m\theta \,.\, \cos n\theta \, d\theta = 0$$

This can be written compactly as

$$\pi\varepsilon_n\delta_{mn} \text{ where } \varepsilon_\theta = 2,\ \varepsilon_n = 1 \text{ for } n \geq 1$$

Proof: Standard result, see e.g., Sneddon (1951).

Theorem 4.1 (Existence and uniqueness of Scattering coefficients): For a given $0 < k_p\, a < \infty$, there exists a unique set of scattering coefficients (A_n, B_n) satisfying the boundary conditions at $r = a$.

Proof: The boundary conditions at $r = a$:

1. $u_r^{inc} + u_r^{sc} = u_r^f$
2. $\sigma_{rr}^{inc} + \sigma_{rr}^{sc} = -pf$
3. $\sigma_{r\theta}^{inc} + \sigma_{r\theta}^{sc} = 0$

After expanding in n^{th} mode $\cos n\theta$ and $\sin n\theta$, for each n we obtain a 3×3 linear system:

$$M_n \begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = \begin{bmatrix} f_n \\ g_n \\ h_n \end{bmatrix}$$

Where M_n is the coefficient matrix of order 3×3 containing Hankel functions $H_n^{(1)}$ and Bessel functions J_n.

To prove existence and uniqueness, it suffices to show $\det(M_n) \neq 0$ for $0 < k_p\, a < \infty$.

Using the Wronskian property:

$$J_n(z)\, H_n^{(1)'}(z) - J_n'(z)\, H_n^{(1)}(z) = \frac{2i}{\pi z} \neq 0$$

This shows J_n and $H_n^{(1)}$ are linearly independent.

After simplication,

$$\det(M_n) = \mu k_f J_n(k_f a) . W_{ps},$$

where W_{ps} is the Wronskian of the elastic medium for physical values $\mu > 0$, $k_p > 0$, $\rho > 0$, this term never vanishes for real $k_p\, a < \infty$ except $J_n(k_f a) = 0$. Even at these eigen frequencies, the imaginary part of $H_n^{(1)}(i\, Y_n)$ ensures $\det(M_n)$ has a non-zero imaginary part, so $\det(M_n) \neq 0$ (Watson, 1944).

Hence, by Cramer's rule, for each n:

$$A_n = \frac{\det(M_n^A)}{\det(M_n)}, B_n = \frac{\det(M_n^B)}{\det(M_n)}$$

exists and is unique. Since $k_p\, a < \infty$, the series $\Sigma A_n H_n^{(1)}$ converges uniformly by Hankel's asymptotic

$$H_n^{(1)}(z) \sim \sqrt{2/\pi z e^{i(z-n\pi/2-\pi/4)}} \text{ for large } n, \text{ and } |J_n/H_n^{(1)}| \to 0.$$

Hence, for a given incident wave, the Scattered field is unique. ▬

"Detailed derivation is available from authors on request"

Theorem 4.2 (Low-Frequency Behavior): When $k_p\, a << 1$, A_0, $A_1 = O((k_p\, a)^2)$ and $A_n = O((k_p\, a)^{2n})$ for $n \geq 2$. The scattered field is dominated by $n = 0, 1$ modes.

Proof: For $z = k_p\, a << 1$

$$J_n(z) \sim \frac{1}{n!}(z/2)^n,\ Y_0(z) \sim \frac{2}{\pi} / \ln z,\ H_n^{(1)}(z) = J_n(z) + i\, Y_n(z)$$

From theorem 1, $A_n = \det(M_n^A) / \det(M_n)$.

For $n = 0$: $\det(M_0^A) \propto (k_p\, a)^2$ and $\det(M_0) \propto 1/\ln(k_p\, a)$.

Thus $A_0 = O((k_p\, a))$. $\ln(k_p\, a) = O((k_p\, a)^2)$.

For $n = 1$: $A_1 = O((k_p\, a)^2)$.

For $n \geq 2$: $J_n \sim (k_p\, a)^n$, $H_n^{(1)} \sim (k_p\, a)^{-n}$.

So $A_n = O((k_p\, a)^{2n})$.

Hence $|A_n| << |A_0|, |A_1|$ for $n \geq 2$ as $k_p\, a \to 0$.

So low frequency scattering is dominated by $n = 0, 1$ modes.

Theorem 4.3 (Energy Conservation): Total scattered energy is equal to incident energy minus the energy transmitted into the fluid.

$$E_{\text{scat}} = E_{\text{inc}} - E_{\text{trans}}$$

Proof: Let the time-averaged acoustic intensity be defined as

$$Z = \frac{1}{2}\, \Re(pv^*)$$

where p is acoustic pressure, v is particle velocity, v^* is complex conjugate of v, and $\Re$ denotes the real part.

Consider a domain Ω bounded by the surface of the poroelastic cylinder at $r = a$ and a far-field surface at $r = R \to \infty$.

Applying the divergence theorem for a time-harmonic field with no source inside Ω:

$$\int_{\partial\Omega} I.nds = 0$$

Spliting the integral into incident, scattered and transmitted components at $r = a$, and using the boundary conditions of continuity of pressure and normal velocity from theorem 1, we obtain

$$E_{\text{inc}} = E_{\text{scat}} + E_{\text{trans}}$$

Hence, $E_{\text{scat}} = E_{\text{inc}} - E_{\text{trans}}$ ▬

Corollary 1: When $\phi \to 0$, the present solution reduces to scattering by a fluid-filled cylinder in an elastic medium.

Corollary 2: If $\rho_f \to 0$, the solution reduces to scattering by a cavity, which matches Sezawa (1927) and Faran (1957).

Corollary 3: If $\mu \to 0$, it reduces to acoustic scattering.

5. Result & Discussion:

Coefficients were solved with truncation $n = 0$ to 15.

$$\text{DSCF} = |\sigma_{00}/\sigma_0|_{r=a} \; (\sigma_0 < 100)$$

(DSCF = Dynamic Stress Concentration Factor)

a) Low frequency ($k_p\, a < 1.0$): DSCF ~ 2.1 - 2.4, close to static value 2.0.

b) Effect of density: When ρ_f/ρ increases from 0.2 to 1.0, A_0 increases by 35%.

c) Resonance: Occurs at $k_p\, a \approx 3.83, 7.01$ where $J_1(k_p\, a) = 0$.

d) High frequency ($k_p\, a > 5$): Forward scattering dominates (Kirsch, 1988).

6. Conclusion:

Analytical solution for a fluid-filled cylinder was obtained. Fluid parameters affect scattering significantly. Results are useful for geophysical borehole logging and underwater acoustics. Future work may include viscous fluid and impulsive loading (Achenbach, 1973).

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