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Norden Metric and Quasi-Kähler Structures

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Abstract:

An almost Norden manifold is a $2n$ -dimensional differentiable manifold M equipped with an almost complex structure J and a Norden metric g , i.e a pseudo-Riemannian metric of neutral signature (n, n) satisfying

$$g(JX, JY) = -g(X, Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

*In this paper, we study a special subclass of almost Norden manifolds, namely **Norden Quasi Manifolds**, characterized by the condition*

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0,$$

*where ∇ denotes the **Levi-civita** connection of g .*

[1]

The main purpose of this article is to obtain characterisations of Norden Quasi-Kähler manifolds in terms of the covariant derivative of the almost complex structure and the curvature tensor of the Norden metric. We derive necessary and sufficient conditions for an almost Norden manifold to be Norden Quasi-Kähler and establish relations between this condition on the base manifold and its complete lift to the tensor bundle $T_s^r M$.

Keywords: Norden metric, Quasi-Kähler manifold, almost Norden structure, Tensor bundle, complete lift, Diagonal lift, Levi-civita connection.

Mathematics Subject Classification 2020: 53C15, 53C25, 53C50, 53B35.

Introduction:

A study of geometric structures on manifolds has been a central theme in modern differential geometry since the fundamental work of **Bernhard Riemann** who introduced the notion of a Riemannian metric in **1854**. Among the various generalisations of Riemannian geometry, the geometry of manifolds equipped with additional compatible structures has attracted considerable attention.

An almost Norden manifold is a $2n$ -dimensional smooth manifold M endowed with an almost complex structure J and a Norden metric g , i.e. a pseudo-Riemannian metric of neutral signature (n, n) satisfying

$$g(JX, JY) = -g(X, Y)$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Such manifolds were systematically studied by **Ganchev** and **Borisov** in **1986** in their classification of almost Norden manifolds.

Gray and **Hervella** classified almost Hermitian manifolds into sixteen classes. One of the important classes in this classification is that of Quasi-Kähler manifolds, characterized by the condition that the covariant derivative of the almost complex structure satisfies

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0.$$

The class of Quasi-Kähler manifolds includes both Kähler and nearly Kähler manifolds and has been extensively studied by **V.F. Kirichenko** and others^[4]. Parallel to this development, the notion of a Norden metric was introduced by the Russian geometer **A.P. Norden (1960)** in the study of almost complex manifolds with a pseudo-Riemannian metric. Unlike the Hermitian case, a Norden metric g satisfies the anti-Hermitian condition

$$g(JX, JY) = -g(X, Y),$$

making it a neutral metric of signature (n, n) . Manifolds equipped with such structures are known as almost Norden manifolds or anti-Hermitian manifolds and have been investigated by **G. Ganchev**, **V. Oproiu** and **A. A. Salimov**.

The interplay between Norden metrics and different classes of almost complex structures leads to a rich geometry. In particular, the concept of a Norden Quasi-Kähler manifold arises when the almost complex structure J satisfies the condition

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0,^{[2]}$$

for all $X, Y \in \mathfrak{X}(M)$. While the metric g is Norden. The study of such manifolds provides a neutral setting for extending results from the Riemannian to pseudo-Riemannian framework and has applications in mathematical physics, especially in the string theory and para-complex geometry.

In this article, we aim to characterise Norden Quasi-Kähler structures. We study the relationship between the Levi-civita connection ∇ of the Norden metric^[3] and the almost complex structure J .

Further, motivated by the work of **K.Yano** and **S. Ishihara** on lifts of geometric objects to tangent bundles, we investigate the behaviour of Norden Quasi-Kähler structures under complete lifts to the tensor bundle $T'_s M$. This approach allows us to establish new characterizations in terms of the complete lift of the almost complex structure and the Levi-civita connection on $T'_s M$, extending the classical results to the tensor bundle setting.

1. Basic Definitions:

Definition 1: Norden metric & almost Norden manifold-

Let M be a $2n$ -dimensional manifold with almost complex structure J , $J^2 = -I$. A pseudo Riemannian metric g of signature (n, n) is called a **Norden metric** or **B. metric**^[5] if

$$g(JX, JY) = -g(X, Y)^{[2]}, \forall X, Y \in \mathfrak{X}(M).$$

The triple (M, J, g) is called almost Norden manifold.

Definition 2: Norden Quasi-Kähler manifold-

An almost Norden manifold (M, J, g) is called **Norden Quasi-Kähler** if the Levi-civita connection ∇ of g satisfies:

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0, \forall X, Y \in \mathfrak{X}(M).$$

Definition 3: Complete lift to tensor Bundle-

For a tensor field K of type $(1, 1)$ on M , its complete lift K^c to $T_s^r M$ is defined by its action on lifted vector fields. For the almost complex structure J , the complete lift J^c satisfies

$$J^c(X^c) = (JX)^c, J^c(X^v) = (JX)^v^{[1]}, \text{ for all } X \in \mathfrak{X}(M),$$

where X^c and X^v denote the complete and vertical lifts of X to $T_s^r(M)$, respectively.

The complete lift J^c satisfies $(J^c)^2 = -I$ on $T_s^r M$. The diagonal lift g^D of Norden metric g gives a Norden metric on $T_s^r M$ with signature $(n(r+s), n(r+s))$.

2. Theorems:

Theorem 1 (Lifting property): Let (M, J, g) be an almost Norden manifold, then $(T_s^r M, J^c, g^D)$ is also an almost Norden manifold.

Proof: We need to show

$$g^D(J^c \tilde{X}, J^c \tilde{Y}) = -g^D(\tilde{X}, \tilde{Y}) \text{ for all } \tilde{X}, \tilde{Y} \in \mathfrak{X}(M).$$

Using properties of complete lift and diagonal lift

1. J^c is almost complex: $(J^c)^2 = -I$
2. For complete lifts X^c, Y^c and vertical lifts X^v, Y^v :

$$g^D(X^c, Y^c) = (g(X, Y)^v, g^D(X^v, Y^v)) \text{ and } J^c(X^c) = (JX)^c, J^c(X^v) = (JX)^v [1]$$

So,

$$g^D(J^c(X^c), J^c(Y^c)) = g^D((JX)^c, (JY)^c)$$

Similarly, we can verify for vertical and mixed terms.

Thus $(T_s^r M, J^c, g^D)$ is almost Norden. ■

Remark: This shows the Norden structure is preserved under complete lift.

Theorem 2 (Quasi-Kähler lift condition): Let (M, J, g) be an almost Norden manifold and let $(T_s^r M, J^c, g^c)$ be its tensor bundle equipped with the complete lift J^c of J and the complete lift g^c of g . Let ∇^c denote the Levi-civita connection of g^c . Then (M, J, g) is a Norden Quasi-Kähler manifold if and only if $(T_s^r M, J^c, g^c)$ is a Norden Quasi-Kähler manifold.

Proof: ($\Rightarrow$) Forward Direction : Assume (M, J, g) is Norden Quasi-Kähler.

$$(1) \quad (\nabla_X J)Y + (\nabla_{JX} J)JY = 0$$

We need to prove the same condition for $(T_s^r M, J^c, g^c)$, i.e.,

$$(\nabla_X^c J^c)Y^c + (\nabla_{J^c X^c}^c J^c)J^c Y^c = 0$$

It is sufficient to check for complete lifts X^c, Y^c and vertical lifts X^v, Y^v , since they span $\mathfrak{X}(T_s^r M)$.

Step 1: Use the following standard formulas for the lifts^[1].

1. $(\nabla_X^c Y^c) = (\nabla_X Y)^c$
2. $J^c(X^c) = (JX)^c, J^c(X^v) = (JX)^v$
3. $(\nabla_X J)Y = \nabla_X JY - J\nabla_X Y$

Step 2: Compute the first term for complete lifts :

$$\begin{aligned}
 (\nabla_{X^c}^c J^c) (Y^c) &= \nabla_{X^c}^c (J^c Y^c) - J^c (\nabla_{X^c}^c Y^c) \\
 &= \nabla_{X^c}^c ((JY)^c) - J^c ((\nabla_X Y)^c) \\
 &= (\nabla_X (JY))^c - (J(\nabla_X Y))^c \\
 &= ((\nabla_X J)Y)^c
 \end{aligned}$$

Step 3: Compute the second term :

$$\begin{aligned}
 (\nabla_{J^c X^c}^c J^c) (J^c Y^c) &= (\nabla_{(JX)^c}^c J^c)(JY)^c \\
 &= ((\nabla_{JX})JY)^c
 \end{aligned}$$

Step 4: Add both terms and use equation (1) :

$$\begin{aligned}
 (\nabla_{X^c}^c J^c) (Y^c) + (\nabla_{J^c X^c}^c J^c)(J^c Y^c) &= ((\nabla_X J)Y)^c + ((\nabla_{JX} J)JY)^c \\
 &= ((\nabla_X J)Y + (\nabla_{JX} J)JY)^c \\
 &= (0)^c = 0.
 \end{aligned}$$

Thus the Quasi-Kähler condition holds for complete lifts. Similar calculation holds for vertical lifts.

Since this holds for complete and vertical lifts, $(T_s^r M, J^c, g^c)$ is Norden Quasi-Kähler.

($\Leftarrow$) Converse Direction:

Suppose $(T_s^r M, J^c, g^c)$ is Norden Quasi-Kähler.

Then for all $\tilde{X} \tilde{Y} \in \mathfrak{X}(T_s^r M)$,

$$(\nabla_{\tilde{X}}^c J^c) \tilde{Y} + (\nabla_{J^c \tilde{X}}^c J^c) J^c \tilde{Y} = 0 \quad [2]$$

In particular, let $\tilde{X} = X^c, \tilde{Y} = Y^c$ be complete lifts.

Using the standard formulas

$$(\nabla_{X^c}^c Y^c) = (\nabla_X Y)^c, J^c(X^c) = (JX^c).^{[1]}$$

Substituting in (2), we get:

$$((\nabla_X J)Y)^c + ((\nabla_{JX} J)JY)^c = 0$$

This implies

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$$

Hence (M, J, g) is Norden Quasi-Kähler. ■

3. Corollaries:

Corollary 1:

Under the assumptions of Theorem 2, the tangent bundle TM admits a Norden Quasi-Kähler structure.

Corollary 2:

Under the assumptions of Theorem 2, the $(l, 1)$ tensor bundle $T_l^1 M$ admits a Norden Quasi-Kähler structure.

These corollaries show that the Norden Quasi-Kähler property is preserved under lifting to tensor bundles^[6].

References:

- [1] Yano, K., Ishihara, S. (1973) : Tangent and cotangent bundle. Differential Geometry. Marcel Dekker, New York.
- [2] Ganchev, G.T., & Borisov, A.T. (1986) : Note on the almost complex manifolds with a Norden metric. C.R. Acad. Bulg. Sci, 39(5), 31-34.
- [3] Staikova, M., & Gribachev, K. (1992) : Canonical Connections and their conformal invariants on weighted manifolds with Norden metric. TS. Geom. Semin. 21, 97-111.

- [4] Cruceanu, V., Fortuny, P. & Gadea, P.M. (1996) : A Survey on para complex geometry. Rocky Mountain Journal of Mathematics, 26(1), 83-115.
- [5] Manev, M., & Gribachev, K. (2008) : On the structure tensors of almost contact manifold with B -metric. Topics in contemporary differential Geometry, Complex Analysis and Mathematical Physics, 117-126.
- [6] Salimov, A.A. (2012) : Tensor operators and their Applications. Nova Science Publishers, New York.
- [7] Lee, J.M. (2018) : Introduction to Riemannian Manifolds. 2nd Ed. Springer, Cham.