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Simplex Method for Solving Optimization Problems having Maximized Objective Functions in Linear Fractional Programming

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Abstract:

In this research paper, we have proposed the simplex method for maximized objective functions in linear fractional programming. By the suitable transformation, one can easily convert the linear fractional programming problem into a linear programming problem. By the solution of linear programming problem thus obtained, we can get an optimal solution of linear fractional programming problem. In the proposed simplex method, the coefficients of the variables under consideration in the transformed linear programming problem are put into the form of a matrix. After applying suitable row transformations, one can easily get the optimal solution of the transformed linear programming problem as well as linear fractional programming problem.

Keywords: fractional programming problem; linear programming problem; objective function; optimal solution; simplex method

1. Introduction:

A Linear Programming Problem (LPP) involves optimizing (maximizing or minimizing) a linear objective function subject to a set of linear equality or inequality constraints. A Linear Fractional Programming Problem (LFPP) deals with optimizing a ratio of two linear functions, typically representing efficiency or cost-benefit scenarios. While LPP is solved using methods like the simplex algorithm, LFPP often requires transformation (e.g., Charnes-Cooper method) to convert it into a standard LPP form. The graphical and simplex methods are two common approaches for solving linear programming problems. When dealing with two-variable problems, the graphical method is simple and effective. However, as the number of variables or constraints increases, especially in complex cases, computational methods like the simplex algorithm become more practical. The simplex method is particularly suitable for computer-based solutions and is widely used to find the optimal value of a linear objective function subject to linear constraints. It is especially useful for solving problems that involve three or more variables.

2. Method:

The simplex technique is a systematic approach for determining the optimal value of a maximization problem. The process involves the following key steps:

1. Construct the initial simplex tableau by introducing slack variables to convert inequalities into equalities.
2. Identify the pivot element:
   (a) Examine the first row to find any negative coefficient, indicating the entering variable.
   (b) For the selected column, compute the ratio of the rightmost column (solution values) to the corresponding column entries.
   (c) Choose the smallest non-negative ratio as it indicates the pivot row.

3. Perform the pivot operation to generate a new tableau.
4. Repeat the process of identifying the pivot and updating the tableau until there are no more negative values in the first row.
5. Interpret the final tableau to obtain the optimal solution.

3. Problem Formulation:

A typical maximization problem is examined in this paper. Prior to solving, the following conditions must be ensured:

(a) The objective function z should be maximized.

(b) All decision variables, $x_1, x_2, x_3,\ldots$, must satisfy non-negativity constraints, i.e., they must be greater than or equal to zero.

(c) Each constraint in the model should be expressed in the form of a "less than or equal to" inequality ($\leq$).

4. Linear-Fractional Programming and Charnes-Cooper Transformation:

A *linear-fractional program* is an extension of the linear program in which the objective function is expressed as a ratio of two affine functions. The optimization is subject to the same type of linear constraints:

$$\begin{aligned} &\text{maximize}_x \quad \frac{c^T x + \alpha}{d^T x + \beta} \\ &\text{subject to} \quad Ax \leq b, \\ &\qquad\qquad\quad\; x \geq 0, \end{aligned}$$

where $d \in \mathbb{R}^n$, and α, β are scalar constants.

The Charnes-Cooper Transformation:

The *Charnes-Cooper transformation* is a variable substitution technique that transforms a linear-fractional program into a linear program. Define new variables:

$$y = \frac{1}{d^T x + \beta} \cdot x, \qquad t = \frac{1}{d^T x + \beta}.$$

Assuming the feasible region is non-empty and bounded, the above linear-fractional program can be rewritten as the equivalent linear program:

$$\begin{aligned} &\text{maximize}_{y,t} \quad c^T y + \alpha t \\ &\text{subject to} \quad Ay \le bt, \\ &\qquad\qquad\quad d^T y + \beta t = 1, \\ &\qquad\qquad\quad t \ge 0. \end{aligned}$$

Once the optimal values of y and t are determined, the original variable x can be recovered using the relation:

$$x = \frac{y}{t}.$$

5. Example: Linear Fractional Programming Problem-

Maximize:

$$z = \frac{6x_1 + 5x_2}{2x_1 + 7}$$

Subject to:

$$\begin{aligned} x_1 + 2x_2 &\le 3, \\ 3x_1 + 2x_2 &\le 6, \\ x_1, x_2 &\ge 0. \end{aligned}$$

Solution: Let $2x_1 + 7 = \frac{1}{y_0}$ and $y = y_0 x$, then

Maximize:

$$z = 6y_1 + 5y_2$$

Subject to:

$$\begin{aligned} y_1 + 2y_2 - 3y_0 &\le 0, \\ 3y_1 + 2y_2 - 6y_0 &\le 0. \end{aligned}$$

Also,

$$2y_1 + 7y_0 = 1, \qquad y_1, y_2, y_0 \geq 0.$$

Simplification:

Maximize:

$$z = 6y_1 + 5y_2$$

$$2y_1 + 7y_0 = 1, \qquad y_1, y_2, y_0 \geq 0.$$

$$y_0 = \frac{1 - 2y_1}{7}$$

Subject to:

$$y_1 + 2y_2 - 3y_0 \leq 0,$$

$$3y_1 + 2y_2 - 6y_0 \leq 0.$$

On substituting the value of y_0 in both equations above and introducing slack variables $y_3, y_4 \geq 0$,

$$13y_1 + 14y_2 + y_3 = 3,$$

$$33y_1 + 14y_2 + y_4 = 6.$$

Simplex Tableau:

$$T_0 = \begin{bmatrix} z & y_1 & y_2 & y_3 & y_4 & b \\ 1 & -6 & -5 & 0 & 0 & 0 \\ 0 & 13 & 14 & 1 & 0 & 3 \\ 0 & 33 & 14 & 0 & 1 & 6 \end{bmatrix}$$

Selection of Pivot:

The most negative indicator in the first row is -5, which corresponds to the column with value 14 in the constraint rows. To determine the pivot row, we compute the test ratios (right-hand side divided by the corresponding entry in the pivot column):

$$\frac{3}{14} \approx 0.214, \quad \frac{6}{14} \approx 0.428$$

Since 0.214 is the smallest non-negative ratio, the pivot element is 14 in the **first constraint row** (i.e., the second row of the tableau).

Elimination by Row Operation - I:

$$R_1 \to R_1 + \frac{5}{14} R_2, \quad R_3 \to R_3 - R_2$$

$$T_1 = \begin{bmatrix} 1 & \frac{-19}{14} & 0 & \frac{5}{14} & 0 & \frac{15}{14} \\ 0 & 13 & 14 & 1 & 0 & 3 \\ 0 & 20 & 0 & -1 & 1 & 3 \end{bmatrix}$$

Pivot column: $\frac{-19}{14} \to$ test ratios:

$$\frac{3}{13} = 0.23, \quad \frac{3}{20} = 0.15 \quad \Rightarrow \quad \text{pivot} = 20$$

Elimination by Row Operation - II:

$$R_1 \to R_1 + \frac{19}{280} R_3, \quad R_2 \to R_2 - \frac{13}{20} R_3$$

Final tableau:

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & \frac{81}{280} & \frac{19}{280} & \frac{357}{280} \\ 0 & 0 & 14 & \frac{33}{20} & \frac{-13}{20} & \frac{21}{20} \\ 0 & 20 & 0 & -1 & 1 & 3 \end{bmatrix}$$

Solution:

$$y_1 = \frac{3}{20}, \quad y_2 = \frac{\frac{21}{20}}{14}, \quad y_3 = 0, \quad y_4 = 0, \quad z = \frac{357}{280}$$

Final Solution:

$$2y_1 + 7y_0 = 1, \quad y_1, y_2, y_0 \geq 0.$$

On putting the value of y_1 in above equation,

$$y_0 = \frac{1}{10}$$

We know, $y = y_0 x$,

$$x_1 = \frac{y_1}{y_0} = \frac{3}{2}, \quad x_2 = \frac{y_2}{y_0} = \frac{3}{4}, \qquad z = \frac{357}{280} = \frac{51}{40}$$

6. Verification of the Solution of the Problem under consideration by different methods:

6.1 Simplex Method for Linear Fractional Programming Problem:

Here we consider an example to explain the simplex procedure:

$$\text{Max } Z = \frac{6x + 5y}{2x + 7}$$

Subject to:

$$x + 2y \leq 3$$

$$3x + 2y \leq 6$$

$$x, y \geq 0$$

Solution: After introducing the slack variables $s_1 \geq 0$ and $s_2 \geq 0$, the given problem becomes in the standard form:

$$\text{Max } z = \frac{6x + 5y}{2x + 7} = \frac{z^{(1)}}{z^{(2)}} \quad \text{(say)}$$

$$x + 2y + s_1 + 0s_2 = 3$$

$$\text{s.t.} \qquad 3x + 2y + 0s_1 + s_2 = 6$$

$$x, y, s_1, s_2 \geq 0$$

By regular simplex method, we get initial table:

$c_j \rightarrow$	6	5	0	0				
$C_j \rightarrow$	2	0	0	0				
Basic Var.	C_B	c_B	x_B	x	y	s_1	s_2	**Min Ratio**
s_1	0	0	3	1	2	1	0	$\frac{3}{1} = 3$
s_2	0	0	6	3	2	0	1	$\frac{6}{3} = 2$
$Z^{(1)} = C_B x_B + \alpha = 0$				-6	-5	0	0	$\leftarrow \Delta_j^{(1)}$
$Z^{(2)} = C_B x_B + \beta = 7$				-2	0	0	0	$\leftarrow \Delta_j^{(2)}$
$Z = \frac{z^{(1)}}{z^{(2)}} = 0$								

Table 1: Initial Simplex Tableau

From the above table, we compute: $Z = \frac{z^{(1)}}{z^{(2)}} = 0$

Now we find:

$$\Delta_j^{(1)} = c_B x_j - c_j \quad \text{and} \quad \Delta_j^{(2)} = C_B x_j - C_j$$

$$\Delta_1^{(1)} = c_B x - c_1 = (0, 0)(1, 3) - 6 = -6,$$

$$\Delta_2^{(1)} = c_B y - c_2 = (0, 0)(2, 2) - 5 = -5,$$

$$\Delta_1^{(2)} = C_B x - C_1 = (0, 0)(1, 3) - 2 = -2,$$

$$\Delta_2^{(2)} = C_B y - C_2 = (0, 0)(2, 2) - 0 = 0,$$

$$\Delta_j = Z^{(2)}(Z_j^{(1)} - c_j) - Z^{(1)}(Z_j^{(2)} - C_j)$$

$$\Delta_1 = (7)(-6) - (0)(-2) = -42$$

$$\Delta_2 = (7)(-5) - (0)(0) = -35.$$

We see that Δ_1 is minimum, hence x is an entering vector.

For the outgoing vector, we calculate:

$$\min\left(\frac{x_{B_i}}{x_{ij}}\right) \text{ when } i = 1;$$

$$\min\left(\frac{x_{B_1}}{x_{1j}}\right) = \min\left(\frac{3}{1}, \frac{6}{3}\right) = 2.$$

This corresponds to x_{21}, so s_2 is the outgoing vector. The key element is 3.

$c_j \rightarrow$	6	5	0	0					
$C_j \rightarrow$	2	0	0	0					
Basic Var.	C_B	c_B	x_B	x	y	s_1	s_2	**Min Ratio**	
s_1	0	0	1	0	$\boxed{\frac{4}{3}}$	1	$-\frac{1}{3}$	$\frac{1}{4/3} = \frac{3}{4}$	
x	2	6	2	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{2}{2/3} = 3$	
$Z^{(1)} = 12$				0	1	0	0	$\leftarrow \Delta_j^{(1)}$	
$Z^{(2)} = 11$				-2/3	4/3	0	0	$\leftarrow \Delta_j^{(2)}$	
$Z = \frac{z^{(1)}}{z^{(2)}} = \frac{12}{11}$					$\uparrow$	$\downarrow$		$\leftarrow \Delta_j$	

Table 2: Second Simplex Tableau

$$\Delta_1^{(1)} = c_B x - c_1 = (0, 6)(0, 1) - 6 = 0,$$

$$\Delta_2^{(1)} = c_B y - c_2 = (0, 6)(0, 1) - 5 = 1,$$

$$\Delta_1^{(2)} = C_B x - C_1 = (0, 2)(4/3, 2/3) - 2 = -2/3,$$

$$\Delta_2^{(2)} = C_B y - C_2 = (0, 2)(4/3, 2/3) - 0 = 4/3,$$

$$\Delta_1 = (11)(0) - (12)(-2/3) = 8$$

$$\Delta_2 = (11)(1) - (12)(4/3) = -5.$$

So Δ_2 is minimum, hence y is an entering vector.

For outgoing vector

$$\min\left(\frac{x_{B_2}}{x_{2j}}\right) = \min\left(\frac{\frac{1}{4}}{\frac{4}{3}}, \frac{\frac{2}{2}}{\frac{2}{3}}\right) = \frac{3}{4}$$

$c_j \rightarrow$	6	5	0	0			
$C_j \rightarrow$	2	0	0	0			
Basic Var.	C_B	c_B	x_B	x	y	s_1	s_2
y	0	5	$\frac{3}{4}$	0	1	$\frac{3}{4}$	$-\frac{1}{4}$
x	2	6	$\frac{3}{2}$	1	0	$-\frac{1}{2}$	$\frac{1}{2}$
$Z^{(1)} = \frac{51}{4}$				0	0	0	0
$Z^{(2)} = 10$				0	0	0	0
$Z = \frac{z^{(1)}}{z^{(2)}} = \frac{51}{40}$							

Table 3: Final Tableau

$$\Delta_1^{(1)} = c_B x - c_1 = (5, 6)(0, 1) - 6 = 0,$$

$$\Delta_2^{(1)} = c_B y - c_2 = (5, 6)(1, 0) - 5 = 0,$$

$$\Delta_1^{(2)} = C_B x - C_1 = (0, 2)(0, 1) - 2 = 0,$$

$$\Delta_2^{(2)} = C_B y - C_2 = (0, 2)(1, 0) - 0 = 0.$$

Since all $\Delta_j \geq 0$, thus the solution is optimal at this level:

$$x = \frac{3}{2}, \quad y = \frac{3}{4}, \quad \max z = \frac{51}{40}.$$

6.2 Solution of problem by Charnes Cooper Transformation Method:

Maximization Problem:

Maximize $z = \dfrac{6x_1 + 5x_2}{2x_1 + 7}$

Subject to:

$$x_1 + 2x_2 \leq 3, \quad 3x_1 + 2x_2 \leq 6, \quad x_1, x_2 \geq 0.$$

Let $2x_1 + 7 = \dfrac{1}{y_0}$ and $y = y_0 x$, then

Maximize $z = 6y_1 + 5y_2$

Subject to:

$$\frac{y_1}{y_0} + 2\frac{y_2}{y_0} \leq 3 \Rightarrow y_1 + 2y_2 - 3y_0 \leq 0,$$

$$3\frac{y_1}{y_0} + 2\frac{y_2}{y_0} \leq 6 \Rightarrow 3y_1 + 2y_2 - 6y_0 \leq 0,$$

$$2x_1 + 7 = \frac{1}{y_0} \Rightarrow 2y_1 + 7y_0 = 1,$$

$$y_1, y_2 \geq 0, \quad y_0 > 0.$$

Now by introducing slack variables $y_3, y_4 \geq 0$ and an artificial variable $w_1 \geq 0$ in the above LPP, and to minimize the infeasibility form $w = w_1$, using the two-phase simplex method we have:

Simplex Table No. 1

Basic Variable	y_0	y_1	y_2	y_3	y_4	w_1	Constant
y_3	-3	1	2	1	0	0	0
y_4	-6	3	2	0	1	0	0
w_1	7	2	0	0	0	1	1
z	0	6	5	0	0	0	0
w	-7↑	-2	0	0	0	-1↓	-1

Simplex Table No. 2

Basic Variable	y_0	y_1	y_2	y_3	y_4	w_1	**Constant**
y_3	0	13/7	2	1	0		3/7
y_4	0	33/7	2	0	1		6/7
y_0	1	2/7	0	0	0		1/7
z	0	6	5	0	0		0
w	0	0↑	0	0	0↓	**drop**	-1

Simplex Table No. 3

Basic Variable	y_0	y_1	y_2	y_3	y_4	w_1	**Constant**
y_3	0	0	40/33	1	-13/33		1/11
y_1	0	1	14/13	0	7/33		2/11
y_0	1	0	-4/33	0	-2/33		1/11
z	0	0	27/11↑	0↓	14/11		12/11

Simplex Table No. 4

Basic Variable	y_0	y_1	y_2	y_3	y_4	w_1	**Constant**
y_2	0	0	1				3/40
y_1	0	1	0				3/20
y_0	1	0	0				1/10
z	0	0	0	81/40	.475		51/40

Optimal Solution:

From the last table, all values of z are positive and $y_0 = 1/10 > 0$, so the optimal solution of the fractional programming problem is:

$$x_1 = \frac{y_1}{y_0} = \frac{\frac{3}{20}}{\frac{1}{10}} = \frac{3}{20} \times 10 = \frac{3}{2}, \quad x_2 = \frac{y_2}{y_0} = \frac{\frac{3}{40}}{\frac{1}{10}} = \frac{3}{40} \times 10 = \frac{3}{4},$$

$$\text{Maximize } z = \frac{6 \times \frac{3}{2} + 5 \times \frac{3}{4}}{2 \times \frac{3}{2} + 7} = \frac{\frac{51}{4}}{10} = \frac{51}{40}.$$

6.3 Solution of Problem by LINGO Software:

LINGO/WIN64 21.0.38 (22 Apr 2025), LINDO API 15.0.6099.233

Licensee info: Eval Use Only

Local optimal solution found.

Objective value:	1.275000
Infeasibilities:	0.000000
Extended solver steps:	5
Best multistart solution found at step:	1
Total solver iterations:	24
Elapsed runtime seconds:	0.14
Model Class:	NLP

Total variables:	2
Nonlinear variables:	2
Integer variables:	0
Total constraints:	5
Nonlinear constraints:	1
Total nonzeros:	8
Nonlinear nonzeros:	2

Variable	Value	Reduced Cost
X1	1.500000	0.000000
X2	0.7500000	0.000000

Row	Slack or Surplus	Dual Price
1	1.275000	1.000000
2	0.000000	0.2025000
3	0.000000	0.4750000E-01
4	1.500000	0.000000
5	0.7500000	0.000000

Figure 1: LINGO Report

The LINGO report provides a computational verification of the results obtained using our proposed method. The solution derived through LINGO aligns perfectly with the outcomes produced by other methods, reinforcing the accuracy and reliability of our approach. This consistency demonstrates the robustness of the method in solving the given optimization problem.

7. Result:

The optimal value is $z = \frac{51}{40}$ · This result represents the solution to the maximization problem, obtained using the simplex method after converting the linear fractional programming problem.

Conclusion:

This research employs the simplex method to address a maximization problem. The simplex approach is a systematic technique for identifying the optimal solution to maximization challenges. It ensures that the solution adheres to the specified constraints while achieving the highest possible value. For the simplex method to be effective, the given problem must first be converted into its standard form. Following the outlined steps in this study enables the derivation of an optimal solution.

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