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Dirichlet Average Associated with Generalized Special Function using Fractional Calculus

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1. Introduction:

There are certain forms of integral averages which are known as Dirichlet averages according to their Dirichlet measure. The theory of multiple types of Dirichlet averaging was introduced by Carlson (1969-1987). Again this has been analyzed by others Mathematicians like Khan et al. (2017), Gupta and Aggarwal (1990-91), Saxena et al. (2010), Sharma and Jain (2006-07), Kilbas and Kattuvetty (2008), Deora and Banerjee (1993), A.K.Thakur et al. (2015), Gurjar (2021). Also a fixed and accurate version of various types of Dirichlet averages was given in a treatise by Carlson. Recently, Meena K; Gurjar M.K. (2020) also found the double Dirichlet average of the 3M parameter multi-index Mittag-Leffler function. In this paper, a study of triple Dirichlet averaging of the Mittag-Leffler function is presented.

2. Generalized Bessel-Maitland Function:

Introduced the famous generalized Bessel-Maitland function and defined it a

$$J_{v,\gamma,\delta}^{\mu,q,p} < z > = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}} \tag{2.1}$$

Where μ, ν, γ, δ ∈ℂ with, $\Re(\nu) > -1$, $\Re(\mu)$, $\Re(\gamma)$, $\Re(\delta) > 0$.

Also, Let $q \in R^+$ with $q < \Re(\mu) + p$.

Where, the generalized Pochhammer is being denoted as $(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}$.

If we take $\vartheta = 0$ and $p = q = \delta = \gamma = 1$ in (2.1) and it is defined as, we get Mittag-Leffler function introduced by Mittag-Leffler G.M.(1903) is received.

$$J_{0,1,1}^{\mu,1,1}(-z) = E_\mu(z)$$

We obtain a generalization of the Bessel-Maitland function presented by Singh et al. (2014). If we take $\delta = p = 1$ in equation (2.1) and it is presented as

$$J_{v,\gamma}^{\mu,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n+v+1)n!} \tag{2.2}$$

Where μ, ν, γ ∈ℂ with, $\Re(\nu) > -1$, $\Re(\gamma)$, $\Re(\mu) > 0$ and $q \in (0,1) \cup N$.

If we put $\nu = \nu - 1$ and $p = q = \delta = \gamma = 1$ in equation (2.1), the function reduced to the well-acknowlaged generalized Mittag - Leffler function introduceded by Wiman A. (2014) as

$$J_{v-1,1,1}^{\mu,1,1}(-z) = E_{\mu,v}(z) \tag{2.3}$$

If we take and define $\vartheta = \vartheta - 1$ and $p = q = \delta = 1$ in (2.1), we get the generalized Mittag - Leffler function given by Prabhakar (1971).

$$J_{v-1,\gamma,1}^{\mu,1,1}(-z) = E_{\mu,v}^{\gamma}(z) \tag{2.4}$$

If we take $\delta = p = 1$ and $q = 0$ and represent it as, we get the Bessel-Maitland function or the Wright generalized function introduced by Marichev.

$$J_v^{\mu}(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\mu n+v+1)n!} = \phi(\mu, \nu + 1; -z) \tag{2.5}$$

If we take and define $\vartheta = \vartheta - 1$ (2.1), we obtain the generalized Mittag-Leffler function given by Sam and Faraz (2007).

$$J_{v-1,\gamma,\delta}^{\mu,q,p}(-z) = E_{\mu,v,p}^{\gamma,\delta,q}(z) \tag{2.6}$$

If we put $\nu = \nu - 1$ and $p = \delta = 1$ in equation (2.1) then the function reduces well to the accepted generalized Mittag-Leffler function introduced by Shukla and Prajapati (2014).

$$J^{\mu,q,1}_{\upsilon-1,\gamma,1}(-z) = E^{\gamma,q}_{\mu,\upsilon}(z) \tag{2.7}$$

3. Mittag-Leffler Function:

Introduced the Mittag-Leffler (p, s, k) function and defined it as

$$_{p}E^{\gamma,s}_{k,\theta,\vartheta}(z) = \sum_{n=0}^{\infty} \frac{p^{(\gamma)_{n,k,s}}}{p^{\Gamma_{s,k}(n\theta+\vartheta)}} \frac{z^n}{n!} \tag{3.1}$$

where $k, p, s \in \Re$, θ, ϑ, $\gamma \in \mathbb{C}$ with $\mathrm{Re}(\theta)$, $\mathrm{Re}(\vartheta)$, $\mathrm{Re}(\gamma) > 0$.

$p^{(\gamma)_{n,k,s}}$ denotes the Pochhammer (p, s, k) symbol which is being defined by Gehlot and Nantomah (2018) as

$$p^{(\gamma)_{n,k,s}} = \left[\frac{\gamma p}{k}\right]_s \left[\frac{\gamma p}{k} + p\right]_s \ldots\ldots\ldots \left[\frac{\gamma p}{k} + (n-1)p\right]_s = \prod_{i=0}^{n-1} \left[\frac{\gamma p}{k} + p\right]_s$$

where $[\gamma]_s = \dfrac{1-s^{\gamma}}{1-s}$, $\forall\, \gamma \in \mathbb{R}$, $0 < s < 1$ and $p^{\Gamma_{s,k}}$ gamma function is defined as

$$p^{\Gamma_{s,k}(\zeta)} = \frac{s}{k} \lim_{n\to\infty} \frac{n!\, p^{n+1} (snp)^{\frac{\gamma}{k}-1}}{p^{(\zeta)_{n,k}}}$$

The relation between three parameters, two parameters and classical Pochhammer's symbol (Ayub, 2020) is given by

$$p^{(\gamma)_{n,k,s}} = s^n p^{(\gamma)_{n,k}} = \left(\frac{sp}{k}\right)^n (\gamma)_{n,k} = (sp)^n \left(\frac{\gamma}{k}\right)_n$$

Also the relation between gamma function of three variables, gamma function of two variables, gamma function of one variables and classical gamma function is given by

$$p^{\Gamma_{s,k}(\zeta)} = (s)^{\zeta/k}\, p^{\Gamma_k(\zeta)} = \left(\frac{sp}{k}\right)^{\zeta/k} \Gamma_k(\zeta) = \frac{(sp)^{\zeta/k}}{k} \Gamma\left(\frac{\zeta}{k}\right)$$

Relation with generalized Wright hypergeometric function

$$_{p}E^{\gamma,s}_{k,\theta,\vartheta}(z) = \frac{k(sp)^{-\vartheta/k}}{\Gamma\left(\frac{\gamma}{k}\right)}\, {}_1\Psi_1 \left[z(sp)^{1-\frac{\theta}{k}} \,\middle|\, \begin{matrix} \left(\frac{\gamma}{k}, 1\right) \\ \left(\frac{\vartheta}{k}, \frac{\theta}{k}\right) \end{matrix} \right] \tag{3.2}$$

Relation with Fox H-function

$$pE^{\gamma,s}_{k,\theta,\vartheta}(z) = \frac{k(sp)^{-\vartheta/k}}{\Gamma\left(\frac{\gamma}{k}\right)} H^{1,1}_{1,2}\left[-z(sp)^{1-\frac{\theta}{k}}\left|\begin{matrix}\left(1-\frac{\gamma}{k},1\right)\\(0,1),\left(1-\frac{\vartheta}{k},\frac{\theta}{k}\right)\end{matrix}\right.\right] \quad (3.3)$$

By inserting $s=1$ in the above equation (3.1), the function reduces to the Mittag-Leffler (P-K) function introduced by Cerutti et al. (as of 2017)

$${}_pE^{\gamma,1}_{k,\theta,\vartheta}(z) = \sum_{n=0}^{\infty}\frac{{}_p(\gamma)_{n,k,1}\, z^n}{{}_p\Gamma_{1,k}(n\theta+\vartheta)\, n!} = {}_pE^{\gamma}_{k,\theta,\vartheta}(z) \quad (3.4)$$

where ${}_p\Gamma_k$ is the p-k-gamma function and ${}_p(\gamma)_{n,k}$ is the p-k-Pochhammer symbol and is defined by Gehlot K.S. (2017)

$${}_p(\gamma)_{n,k} = \left(\frac{\gamma p}{k}\right)\left(\frac{\gamma p}{k}+p\right)\left(\frac{\gamma p}{k}+2p\right)\dots\left(\frac{\gamma p}{k}+(n-1)p\right) = \frac{{}_p\Gamma_k(\gamma+nk)}{{}_p\Gamma_k(\gamma)}$$

and ${}_p\Gamma_k$ is defined as (p-k) gamma function

$${}_p\Gamma_k(z) = \int_0^{\infty} t^{z-1}e^{-\frac{t^k}{p}}\,dt \quad (z \in C/k\mathbb{Z}^-; p,k \in \mathcal{R}^+/\{0\})$$

Relation with the generalized Wright hypergeometric function $p\Psi q$

$$pE^{\gamma}_{k,\theta,\vartheta}(z) = \frac{k(p)^{-\vartheta/k}}{\Gamma\left(\frac{\gamma}{k}\right)}\, {}_1\Psi_1\left[z(p)^{1-\frac{\theta}{k}}\left|\begin{matrix}\left(\frac{\gamma}{k},1\right)\\\left(\frac{\vartheta}{k},\frac{\theta}{k}\right)\end{matrix}\right.\right] \quad (3.5)$$

Relationship with Fox H-function

$$pE^{\gamma}_{k,\theta,\vartheta}(z) = \frac{k(p)^{-\vartheta/k}}{\Gamma\left(\frac{\gamma}{k}\right)} H^{1,1}_{1,2}\left[-z(p)^{1-\frac{\theta}{k}}\left|\begin{matrix}\left(1-\frac{\gamma}{k},1\right)\\(0,1),\left(1-\frac{\vartheta}{k},\frac{\theta}{k}\right)\end{matrix}\right.\right] \quad (3.6)$$

Putting $p=1$ and $s=1$ in (3.1), the function converges to the Mittag-Leffler K-function given by Dorrego and Cerutti (2017) and is defined as

$${}_1E^{\gamma,1}_{k,\theta,\vartheta}(z) = \sum_{n=0}^{\infty}\frac{{}_1(\gamma)_{n,k,1}\, z^n}{{}_1\Gamma_{1,k}(n\theta+\vartheta)\, n!} = \sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}\, z^n}{\Gamma_k(n\theta+\vartheta)\, n!} = E^{\gamma}_{k,\theta,\vartheta}(z). \quad (3.7)$$

where Γ_k is the k-gamma function and $(\gamma)_{n,k}$ is the Pochhammer k-symbol and is defined by Diaz and Pariguán (2007)

$$(\gamma)_{n,k} = \gamma(\gamma+k)(\gamma+2k)\dots\dots(\gamma+(n-1)k)$$

And $\Gamma_k(z)$ is defined as k-gamma function

$$\Gamma_k(z) = \int_0^{\infty} t^{z-1} e^{-\frac{t^k}{k}} dt \qquad (\Re e(z) > 0)$$

Generalized right hypergeometric function and relation with Fox H-function

$$E^{\gamma}_{k,\theta,\vartheta} = \frac{k}{\Gamma\left(\frac{\gamma}{k}\right)} {}_1\Psi_1 \left[z \,\middle|\, \begin{matrix} \left(\frac{\gamma}{k}, 1\right) \\ \left(\frac{\vartheta}{k}, \frac{\theta}{k}\right) \end{matrix} \right] = \frac{k}{\Gamma\left(\frac{\gamma}{k}\right)} H^{1,1}_{1,2} \left[-z \,\middle|\, \begin{matrix} \left(1-\frac{\gamma}{k}, 1\right) \\ (0,1), \left(1-\frac{\vartheta}{k}, \frac{\theta}{k}\right) \end{matrix} \right] \quad (3.8)$$

If we substitute k=1, p=1 and s=1 and determine the following function then equation (3.3.1). Prabhakar T.R. Reduces to the Mittag-Leffler function provided by (1971)

$${}_1E^{\gamma,1}_{1,\theta,\vartheta}(z) = \sum_{n=0}^{\infty} \frac{{}_1(\gamma)_{n,1,1}\, z^n}{{}_1\Gamma_{1,1}(n\theta+\vartheta)\, n!} = \sum_{n=0}^{\infty} \frac{(\gamma)_n\, z^n}{\Gamma(n\theta+\vartheta)\, n!} = E^{\gamma}_{\theta,\vartheta}(z). \qquad (3.9)$$

where $(\gamma)_n$ = denotes the pochhammer symbol

$$(\gamma)_n = \gamma(\gamma+1)(\gamma+2)\ldots..(\gamma+(n-1)) = \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)}$$

Generalized right hypergeometric function and relation with Fox H-function

$$E^{\gamma}_{\theta,\vartheta}(z) = \frac{1}{\Gamma(\gamma)} {}_1\Psi_1 \left[z \,\middle|\, \begin{matrix} (\gamma, 1) \\ (\vartheta, \theta) \end{matrix} \right] = \frac{1}{\Gamma(\gamma)} H^{1,1}_{1,2} \left[-z \,\middle|\, \begin{matrix} (1-\gamma, 1) \\ (0,1), (1-\vartheta, \theta) \end{matrix} \right] \quad (3.10)$$

By setting k=1, p=1, s=1 and γ=1 in equation (3.1), the function is generated in the Mittag-Leffler function introduced by Wiman [1905]

$${}_1E^{1,1}_{1,\theta,\vartheta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\theta+\vartheta)} = E_{\theta,\vartheta}(z). \qquad (3.11)$$

Generalized right hypergeometric function and relation with Fox H-function

$$E_{\theta,\vartheta}(z) = {}_1\Psi_1 \left[z \,\middle|\, \begin{matrix} (1,1) \\ (1, \theta) \end{matrix} \right] = H^{1,1}_{1,2} \left[-z \,\middle|\, \begin{matrix} (0,1) \\ (0,1), (1-\vartheta, \theta) \end{matrix} \right] \qquad (3.12)$$

Similarly, In the equation (3.1), the function reduces to the so called Mittag-Leffler function presented by Mittag-Leffler G.M. (1903). If we substitute k=1, p=1, s=1, ϑ=1 and γ=1 the following function are determind

$${}_1E^{1,1}_{1,\theta,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\theta+1)} = E_{\theta}(z) \qquad (3.13)$$

Generalized right hypergeometric function and relation with Fox H-function

$$E_\theta(z) = {}_1\Psi_1\left[z \,\middle|\, \begin{matrix}(1,1)\\(1,\mathrm{k})\end{matrix}\right] = H^{1,1}_{1,2}\left[-z \,\middle|\, \begin{matrix}(0,1)\\(0,1),(0,\theta)\end{matrix}\right] \tag{3.14}$$

further, we obtain exponential function when we take $k=1, p=1, s=1, \vartheta=1, \gamma=1$ and $\theta=1$ in equation (3.1).

4. Dirichlet Average:

The theory of Dirichlet averaging, Set $z=(z_1,\dots,z_k)\in\Im^k$, $k\geq 3$ and take $\Im$ as a convex set in the complex numbers $\mathbb{C}$, where f is a measurable function on $\Im$.

Then we get

$$F(b;z)=\int_E f(u \circ z)\, d\mu_b(u) \tag{4.1}$$

Here the equation (4.1) explains $d\mu_b(x)$ and

$$u \circ z=\sum_{i=1}^{k} u_i z_i \text{ and } u_n=1-u_1-\dots-u_{k-1}$$

Triple Dirichlet Average:

The Triple Dirichlet average is analogous to the fractional derivative. Let there be a $k\times x$ matrix with complex elements z_{ijk} on the setting. Let $u=(u_1,\dots,u_k)$, $v=(v_1,\dots,v_x)$, and $w=(w_1,\dots,w_y)$ ordered k-tuple, x-tuple and y-tuple of real non-negative weights $\sum u_1=1$, $\sum v_j=1$ and $\sum w_k=1$ respectively.

Now we define

$$u \circ z \circ v \circ w=\sum_{i=1}^{x}\sum_{j=1}^{y}\sum_{k=1}^{z} u_i z_{ijk} v_j w_k$$

every convex combinations becomes points in the convex hull $H(z)$ stands form of $(z_{111},\dots,z_{kxy})$. If z_{ijk} is considered as a point of the complex plane.

By taking $b=(b_1,\dots,b_k)$ be an ordered k-tuple of complex number with positive real part $\mathrm{Re}(b)>0$ and then as well for $\beta=(\beta_1,\dots,\beta_x)$, and $\gamma=(\gamma_1,\dots,\gamma_y)$, then explain $d\mu_b(u)$, $d\mu_\beta(v)$ and $d\mu_\gamma(w)$.

Suppose the holomorphic on a domain D in the complex plane is f, if $\mathrm{Re}(b)>0$, $\mathrm{Re}(\beta)>0$, $\mathrm{Re}(\gamma)>0$ and $H(z)\subset D$. Then we obtain

$$F(b, z, \beta, \gamma) = \iiint f(u \circ z \circ v \circ w) d\mu_b(u) d\mu_\beta(v) d\mu_\gamma(w)$$

Gupta and Agrawal (1975) presented the S-function $(x \circ z \circ y)^t$ with a Double Average for $(z=x=y=2)$ under

$$S(\mu_1, \mu_{2:}\mu_3; z; \rho_1, \rho_2, \rho_3; \sigma_1, \sigma_2, \sigma_3)$$

$$= \int_0^1 \int_0^1 \int_0^1 (x\ ozoy)^t\ dm_{\mu_1,\mu_{2:}\mu_3}(\mathrm{u}) dm_{\rho_1,\rho_2,\rho_3}(\mathrm{v})\ \ dm_{\sigma_1,\sigma_2,\sigma_3}(\mathrm{w}) \quad (4.2)$$

Where $\mathrm{Re}(\mu_1)>0$, $\mathrm{Re}(\mu_2)>0$, $\mathrm{Re}(\mu_3)>0$, $\mathrm{Re}(\rho_1)>0$, $\mathrm{Re}(\rho_2)>0$, $\mathrm{Re}(\rho_3')>0$, $\mathrm{Re}(\sigma_1)>0$, $\mathrm{Re}(\sigma_2)>0$, $\mathrm{Re}(\sigma_3)>0$ and

$$\mathrm{u\ o}\ zo\ vow = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}(u_i \text{ o } z_{ijk}\ \text{ o}v_j \text{o } \mathrm{w_k}$$

$$= \sum_{i=1}^{2}[u_1(z_{i1}v_1 + z_{i2}v_2 + z_{i2}w_2)]$$

$$= \ [u_1 z_{11} v_1 w_1 + u_1 z_{12} v_2 w_2 + u_2 z_{21} v_1 w_1 + u_2 z_{22} v_2 w_2]$$

Suppose $z_{11} = a$, $z_{12} = z_{21} = b$, $z_{13} = z_{31} = c$ and $z_{21} = d$, $z_{22} = e$, $z_{23} = f$ and

$$\begin{cases} u_1 = u, & u_2 = 1-u \ , & u_3 = 1-u \\ v_1 = v, & v_2 = 1-v\ , & v_3 = 1-v \\ w_1 = w, & w_2 = 1-w\ , & w_3 = 1-w \end{cases}$$

$$\text{Thus } z = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$$

Therefore

$$u \circ z \circ v \circ w = uvwab + uvb(1-v)(1-w) + (1-u)(1-w)cv + (1-u)d(1-v)(1-w) \quad (4.3)$$

$$= uvw(a-b-c-d-e+f) + u(b-d-f) + v(c-d-a-f) + w(d-e-f-a) + d$$

$$dm_{\mu_1\mu_{2,}\mu_3}(\mathrm{u}) \ = \ \frac{\Gamma(\mu_1+\mu_2+\mu_3)}{\Gamma(\mu_1)\Gamma\mu_2)\Gamma\mu_3)}\ u^{\mu_1-1}\,(1-u)^{\mu_2-1}(1-u)^{\mu_3-1}\,du \quad (4.4)$$

$$dm_{\rho_1\rho_{2,}\rho_3}(\mathrm{v}) \ = \ \frac{\Gamma(\rho_1+\rho_2+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)}\ v^{\rho_1-1}\,(1-v)^{\rho_2-1}\,(1-v)^{\rho_3-1}d\mathrm{v} \quad (4.5)$$

and $$dm_{\sigma_1\sigma_{2,}\sigma_3}(\mathrm{v}) = \frac{\Gamma(\sigma_1+\sigma_2+\sigma_2)}{\Gamma(\sigma_1)\Gamma(\sigma_2)\Gamma(\sigma_3)} v^{\sigma_1-1}(1-w)^{\sigma_2-1}(1-w)^{\sigma_3-1}\mathrm{dv} \tag{4.6}$$

Thus from equation (4.6), we obtain

$$\mathrm{S}(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3;\sigma_1\sigma_{2,}\sigma_3)$$

$$= \frac{\Gamma(\mu_1+\mu_2+\mu_3)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)} \frac{\Gamma(\rho_1+\rho_2+\rho_3)\Gamma(\sigma_1+\sigma_2+\sigma_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)\Gamma(\sigma_1)\Gamma(\sigma_2)\Gamma(\sigma_3)}$$

$$\times \int_0^1\int_0^1\int_0^1 [uvw(a-b-c-d-e+f)+uv(b-d-f)+vw(c-d-e)+f]^t$$

$$\times u^{\mu_1-1}(1-u)^{\mu_2-1}(1-u)^{\mu_3-1}\mathrm{du}\ v^{\rho_1-1}(1-v)^{\rho_2-1}(1-v)^{\rho_3-1}\mathrm{dv}$$

$$\times w^{\sigma_1-1}(1-w)^{\sigma_2-1}(1-w)^{\sigma_3-1}\mathrm{du} \tag{4.7}$$

5. Fractional Derivative:

The Riemann-Liouville fractional integral of arbitrary order δ Re(δ) > 0

$$\left(I_{a+}^{\delta}\,\mathrm{F}\right)(\mathrm{z}) = \frac{1}{\Gamma(\delta)}\int_a^z (z-s)^{\delta-1}F(s)\ ds \quad z>a, a\in \mathrm{R}.$$

We obtain the fractional derivative by proceeding through fractional integration. Erdélyi et al. (1954) explained the Riemann-Liouville fractional differential operator D_s^{δ} of order $\delta(\delta\in\mathbb{C})$ as

$$\left(D_z^{\delta}\mathrm{F}\right)(\mathrm{z}) = \begin{cases} \dfrac{1}{\Gamma(-\delta)}\displaystyle\int_a^z (z-s)^{-\delta-1}\ \mathrm{F}(s)\mathrm{d}s; & \Re e(\delta) < 0 \\ \dfrac{d^n}{dz^n}\{D_z^{\delta-n}(f(z))\}; & n-1 \le \Re(\delta) < n; n\in N. \end{cases} \tag{5.1}$$

Here $z^p(z)$ is as $F(z)$ and Re(δ) < 0; is analytic for $z = 0$.

6. Dirichlet Average of Generalized Bessel-Maitland Function and Mittag-Leffler Function:

In this section, we develop the Dirichlet average of generalized Bessel Maitland function and Mittag-Leffler function

Theorem 1: Let μ, ν, ω, γ, δ, ε∈C with $\Re(\mu) > 0$, $\Re(\nu) > -1$, $\Re(\omega) > -1$, $\Re(\gamma) > 0$, $\Re(\delta)$, $\Re(\varepsilon) > -1$. Also, Let $p, q, r \in R^+$ with $q < \Re(\mu) + p$. Then the function defined the triple Dirichlet average of generalized Bessel-Maitland function $J^{\mu,q,p,r}_{\nu,\omega,\gamma,\delta,\varepsilon}(uozovow)$ involving the fractional derivative for ($k=x=y=2$) as

$$S(\mu_1, \mu_2\mu_3; z; \rho_1, \rho_2, \rho_3)$$
$$= \frac{\Gamma(\rho_1 + \rho_2 + \rho_3)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2-\rho_3}\left[D^{-\rho_3}_{x-y}\left\{t^{\rho_1-1}J^{\mu,q,p,r}_{v,\gamma,\delta,\varepsilon}(y+t)\right\}\right](x-y)$$

Proof: Let take the triple Dirichlet average for ($k=y=x=2$) of generalized Bessel-Maitland function $J^{\mu,q,p,r}_{\nu,\omega,\gamma,\delta,\varepsilon}(uozovow)$ with the help of equation (4.6). Let us take the double Dirichlet average for ($k=x=y=2$) of the generalized Bessel-Maitland function $J^{\mu,q,p,r}_{\nu,\omega,\gamma,\delta,\varepsilon}$ with the help of equation (4.6)

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3) = \int_0^1\int_0^1\int_0^1 J^{\mu,q,p}_{v,\gamma,\delta}(u \circ z \circ v)\, dm_{\mu_1\mu_2}(u) dm_{\rho_1\rho_2}(v) \quad (6.1)$$

Interpreting equation (6.1) and changing the order of integration and summation we get the following equation

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3)$$
$$= \sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n + v + 1)(\delta)_{pn}}\int_0^1\int_0^1\int_0^1 (u \circ z \circ v)^n\, dm_{\mu_1\mu_{2,}\mu_3}(u)\;\; dm_{\rho_1\rho_{2,}\rho_{3,}}(v)$$

applying (4.11), (4.12) and (6.1), we have

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3)$$
$$= \frac{\Gamma(\mu_1 + \mu_2 + \mu_3)}{\Gamma(\mu_1)\Gamma(\mu_2)}\frac{\Gamma(\rho_1 + \rho_2 + \rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n + v + 1)(\delta)_{pn}}$$
$$\times \int_0^1\int_0^1\int_0^1 [uv(a-b-c+d) + u(b-d) + v(c-d) + d]$$
$$\times\, u^{\mu_1-1}(1-u)^{\mu_2-1}(1-u)^{\mu_3-1}du v^{\rho_1-1}(1-v)^{\rho_2-1}(1-v)^{\rho_3-1}dv \quad (6.2)$$

Putting $a=c=d=x$, $e=f=y$, then we obtain

$$S(\mu_1, \mu_2 \mu_3; z; \rho_1, \rho_2, \rho_3) = \frac{\Gamma(\mu_1+\mu_2+\mu_3)}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$

$$\times \int_0^1 \int_0^1 \int_0^1 [v(x-y)+y]^n u^{\mu_1-1}(1-u)^{\mu_2-1}(1-u)^{\mu_3-1} du\, v^{\rho_1-1}(1-v)^{\rho_2-1}(1-v)^{\rho_3-1} dv$$

Furthermore, with the help of the well-known beta and gamma functions, we get

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3) = \frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$

$$\times \int_0^1 [v(x-y)+y]^n v^{\rho_1-1}(1-v)^{\rho_2-1}(1-v)^{\rho_3-1} dv$$

Let $v(x-y) = t$, then we have

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3) = \frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-z)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$

$$\times \int_0^{x-y} (y+t)^n \left(\frac{t}{x-y}\right)^{\rho_1-1} \left(1-\frac{t}{x-y}\right)^{\rho_2-1} \left(1-\frac{t}{x-y}\right)^{\rho_3-1} \frac{dt}{x-y}$$

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3) = \frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)} \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}} (x-y)^{1-\rho_1-\rho_2-\rho_3}$$

$$\times \int_0^{x-y} t^{\rho_1-1}(y+t)^n (x-y-t)^{\rho_2-1}(x-y-t)^{\rho_3-1} dt$$

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3) = \frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)} (x-y)^{1-\rho_1-\rho_2-\rho_3}$$

$$\int_0^{x-y} (x-y-t)^{\rho_2-1} \left\{ t^{\rho_1-1} J_{v,\gamma,\delta,\varepsilon}^{\mu,q,p,r}(y+t) \right\} dt$$

Utilizing the explaination of fractional derivative given in (4.2), we obtain needed outcome.

Corollaries:

1. Let $\delta = p = 1$ fulfill the condition of theorem 1 then we obtain the following results hold:

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3)$$
$$= \frac{\Gamma(\rho_1 + \rho_2 + \rho_3)}{\Gamma(\rho_1)}(x - y)^{1-\rho_1-\rho_2-\rho_3}\left[D_{x-y}^{-\rho_2}\{t^{\rho_1-1}J_{v,\gamma}^{\mu,q}(y + t)\}\right](x - y)$$

2. Let $v = v - 1$ and $\delta = p = q = 1$ fulfill the condition of theorem 1 then we obtain the following results hold

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3)$$
$$= \frac{\Gamma(\rho_1 + \rho_2 + \rho_3)}{\Gamma(\rho_1)}(x - y - z)^{1-\rho_1-\rho_2-\rho_3}\left[D_{x-y}^{-\rho_2}\{t^{\rho_1-1}E_{\mu,v}^{\gamma}(y + t)\}\right](y - x - z)$$

which is well known result by Khan et al. (2017).

Theorem 2: Let $\mu, \nu, \gamma, \delta \in \mathbb{C}$ with $\Re(\nu) > -1$, $\Re(\mu)$, $\Re(\gamma)$, $\Re(\delta)$, $\Re(\varepsilon) > 0$. Also, Let $p, q \in R^+$ with $q < \Re(\mu) + p$. Then the function defined the triple Dirichlet average of generalized Bessel-Maitland function $J_{\nu,\omega,\gamma,\delta,\varepsilon}^{\mu,q,p,r}(uozovow)$ involving the fractional derivative for ($k = y = x = z = 2$) as

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3)$$
$$= \frac{(\mu_1)_n}{(\mu_1 + \mu_2 + \mu_3)_n}\frac{\Gamma(\rho_1 + \rho_2 + \rho_3)}{\Gamma(\rho_1)}(x - y - z)^{1-\rho_1-\rho_2}$$
$$\left[D_{x-y}^{-\rho_2}\left\{t^{\rho_1-1}J_{v,\gamma,\delta}^{\mu,q,p}(y + t)\right\}\right](x - y - z)$$

Proof: Using (4.2), we have

$$S(\mu_1, \mu_2, \mu_3; z; \rho_1, \rho_2, \rho_3)$$
$$= \frac{\Gamma(\mu_1 + \mu_2 + \mu_3)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)}\frac{\Gamma(\rho_1 + \rho_2 + \rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n + v + 1)(\delta)_{pn}}$$

$$\times \int_0^1 \int_0^1 \int_0^1 [uv(a-b-c+d)+u(b-d)+v(c-d)+d]^n$$

$$\times u^{\mu_1-1}(1-u)^{\mu_2-1}du \; v^{\rho_1-1}(1-v)^{\rho_2-1}dv$$

We suppose that $a=x$, $b=y$, $c=d=e=f=0$, then obtain

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)$$

$$=\frac{\Gamma(\mu_1+\mu_2+\mu_3)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)}\frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$

$$\times \int_0^1 \int_0^1 \int_0^1 [uvw(x-y)+uy]^n u^{\mu_1-1}(1-u)^{\mu_2-1}(1-u)^{\mu_3-1}v^{\rho_1-1}$$

$$(1-v)^{\rho_2-1}(1-v)^{\rho_3-1}w^{\rho_1-1}(1-w)^{\rho_2-1}(1-w)^{\rho_3-1}du\,dvdw$$

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)$$

$$=\frac{\Gamma(\mu_1+\mu_2+\mu_3)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)}\frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$

$$\times \int_0^1 \int_0^1 \int_0^1 [vx+uy+z(1-w)+z(1-w)]^n u^{\mu_1+n-1}(1-u)^{\mu_2-1}$$

$$(1-u)^{\mu_3-1}(1-v)^{\rho_2-1}(1-v)^{\rho_3-1}\; du\,dvdw$$

Also, with the help of well known Beta and Gamma functions, we get

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)$$

$$=\frac{\Gamma(\mu_1+\mu_2+\mu_3+\mu_3)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)}\frac{\Gamma(\rho_1+\rho_2+\rho_3)\Gamma(\mu_1+n)\Gamma(\mu_2)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)\Gamma(\mu_1+\mu_2+n)}$$

$$\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}\times\int_0^1[v(x-y-z)+y]^n v^{\rho_1-1}(1-v)^{\rho_2-1}dv$$

on putting $v(x-y-z)=t$ in above equation, then we get

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)$$

$$=\frac{\Gamma(\rho_1+\rho_2+\rho_3)(\mu_1)_n}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)(\mu_1+\mu_2+\mu_3)_n}\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}(-1)^n}{\Gamma(\mu n+v+1)(\delta)_{pn}}$$

$$\times\int_0^{x-y-z}(y+t)^n\left(\frac{t}{x-y-z}\right)^{\rho_1-1}\left(1-\frac{t}{x-y-z}\right)^{\rho_2-1}$$

$$\left(1-\frac{t}{x-y-z}\right)^{\rho_3-1}\frac{dt}{x-y-z}$$

Utilizing the explanation of fractional derivative given in (4.2), we obtain needed outcome

Special Cases:

1. Let $\delta=p=q=1$ and $v=v-1$ fulfill the condition of theorem 2 then we obtain the following results hold:

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)=\frac{(\mu_1)_n}{(\mu_1+\mu_2+\mu_3)_n}\frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}$$

$$-\rho_3\left[D_{x-y}^{-\rho_2}\{t^{\rho_1-1}E_{\mu,v}^{\gamma}(y+t)\}\right](y-x)$$

Which is well known result by Khan et. al. (2017).

Theorem 3: Let $p,k,s,r\in\mathbb{N},\theta,\vartheta,\gamma\in\mathbb{C}$ with $\Re e(\theta)>0$, $\Re e(\vartheta)>0$, $\Re e(\gamma)$, $\Re(\varepsilon)>0$, then the triple Dirichlet average of Mittag-Leffler function $pE_{v,\omega,\gamma,\delta,\varepsilon}^{\mu,q,p,r}(uozovow)$ involving the fractional derivative for $(k=x=y=2)$ is

$$S(\mu_1,\mu_2,\mu_2;z;\rho_1,\rho_2,\rho_3)$$

$$=\frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2-\rho_3}\left[D_{u-v}^{-\rho_2}\{t^{\rho_1-1}pE_{k,\theta,\vartheta}^{\gamma,s,r}(y+t)\}\right](x-y)$$

$$=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\frac{k(srp)^{\vartheta/k}}{\Gamma\left(\frac{\gamma}{k}\right)}(x-y)^{1-\rho_1-\rho_2-\rho_3}$$

$$\times\left(D_{u-v}^{-\rho_3}\left\{t^{\rho_1-1}{}_1\Psi_1\left[\begin{matrix}\left(\frac{\gamma}{k},1\right)\\ \left(\frac{\vartheta}{k},\frac{\theta}{k}\right)\end{matrix};(y+t)(srp)^{1-\frac{\theta}{k}}\right]\right\}\right)(x-y)$$

$$=\frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)}\frac{k(srp)^{\vartheta/k}}{\Gamma\left(\frac{\gamma}{k}\right)}(x-y)^{1-\rho_1-\rho_2}$$

$$\times\left(D_{u-v}^{-\rho_2}\left\{t^{\rho_1-1}H_{1,2}^{1,1}\left[\begin{matrix}\left(1-\frac{\gamma}{k},1\right)\\ (0,1)\ \left(1-\frac{\vartheta}{k},\frac{\theta}{k}\right)\end{matrix};-(y+t)(srp)^{1-\frac{\theta}{k}}\right]\right\}\right)(x-y)$$

Proof: Let take the double Dirichlet average for (k=x=y=2) of Mittag-Leffler function $pE_{k,\theta,\vartheta}^{\rho,s,r}(u\circ z\circ v)$ into consideration with the help of equation (4.2)

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)=\int_0^1\int_0^1\int_0^1\{pE_{k,\theta,\vartheta}^{\gamma,s,r}(u\circ z\circ vow)\}\,dm_{\mu_1\mu_2}(u)dm_{\rho_1\rho_2}(v) \tag{4.15}$$

We get the following equation interpreting on equation (4.15) and interchanging the order of integration and summation

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)=\sum_{n=0}^{\infty}\frac{p^{(\gamma)}{}_{n,k,s,r}}{p^{\Gamma_{r,s,k}(n\theta+\vartheta)}}\frac{z^n}{n!}\int_0^1\int_0^1\int_0^1((u\circ z\circ vow))^n$$

$$dm_{\mu_1\mu_2\mu_2\mu_3}(u)\,dm_{\rho_1\rho_2\rho_3}(v)$$

Applying (4.18), (4.19)and (4.20) , we have

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)$$

$$=\frac{\Gamma(\mu_1+\mu_2+\mu_3)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\mu_3)}\frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\rho_3)}\sum_{n=0}^{\infty}\frac{p^{(\gamma)}{}_{n,k,s,r}}{p^{\Gamma_{s,k}(n\theta+\vartheta)}}$$

$$\times\int_0^1\int_0^1\int_0^1[uv(a-b-c+d)+u(b-d)+v(c-d)+d]^n$$

$$\times u^{\mu_1-1}(1-u)^{\mu_2-1}du\ v^{\rho_1-1}(1-v)^{\rho_2-1}dv$$

We suppose that $a=c=x$, $b=d=y$, then we obtain

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)$$
$$=\frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\frac{\Gamma(\beta_1+\beta_2)}{\Gamma(\beta_1)\Gamma(\beta_2)}\sum_{n=0}^{\infty}\frac{p^{(\gamma)}{}_{n,k,s}}{p^{\Gamma_{s,k}(n\theta+\vartheta)}}\int_0^1\int_0^1[\nu(x-y)+\nu]^n$$
$$\times u^{\mu_1-1}(1-u)^{\mu_2-1}\,\mathrm{du}\ \nu^{\rho_1-1}(1-\nu)^{\rho_2-1}\,\mathrm{d}\nu$$

Also, with the help of well-known Beta and Gamma function, we get

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)=\int_0^1[\nu(x-y)+\nu]^n\,\nu^{\rho_1-1}(1-\nu)^{\rho_2-1}\,d\nu$$

Suppose $\nu(x-y)=t$ then we have

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\sum_{n=0}^{\infty}\frac{p^{(\gamma)}{}_{n,k,s}}{p^{\Gamma_{s,k}(n\theta+\vartheta)}}$$
$$\times\int_0^{u-v}(y+t)^n\left(\frac{t}{x-y}\right)^{\rho_1-1}\left(1-\frac{t}{x-y}\right)^{\rho_2-1}\frac{dt}{x-y}$$
$$=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}(x-y)^{1-\rho_1-\rho_2}\int_0^{x-y}(x-y-t)^{\rho_2-1}$$
$$\{t^{\rho_1-1}\,pE^{\gamma,s}_{k,\theta,\vartheta}(y+t)\}dt$$

Utilising the explanation of fractional derivative given in we obtain needed outcome

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)$$
$$=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y-z)^{1-\rho_1-\rho_2}\left[D^{-\rho_2}_{u-v}\{t^{\rho_1-1}pE^{\gamma,s}_{k,\theta,\vartheta}(y+t)\}\right](x-y-z)$$

Special Cases:

1. Let $s=1, r=1, p=1, k=1$ fulfill the condition of theorem 1, then the following result holds:

$$(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)=\frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)}(x-y-z)^{1-\rho_1-\rho_2-\rho_3}$$
$$\left[D^{-\rho_2}_{x-y}\{\tau^{\rho_1-1}E^{\gamma}_{\theta,\vartheta}(y+t)\}\right](x-y-z)$$

$$= \frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)} \frac{1}{\Gamma(\gamma)} (x-y-z)^{1-\rho_1-\rho_2}$$

$$\times \left(D_{x-y}^{-\rho_2} \left\{ \tau^{\rho_1-1} {}_1\Psi_1 \left[\begin{matrix} (\gamma,1) \\ (\vartheta,\theta) \end{matrix}; (y+t) \right] \right\} \right) (x-y-z)$$

$$= \frac{\Gamma(\rho_1+\rho_2+\rho_3)}{\Gamma(\rho_1)} \frac{1}{\Gamma(\gamma)} (x-y-z)^{1-\rho_1-\rho_2-\rho_3}$$

$$\times \left(D_{x-y}^{-\rho_2} \left\{ \tau^{\rho_1-1} H_{1,1,2}^{1,1,1} \left[\begin{matrix} (1-\gamma,1) \\ (1-\vartheta,\theta) \end{matrix}; -(y+t) \right] \right\} \right) (x-y-z)$$

which is well known result by Gurjar M.K.(2020).

2. Let $s=1, p=1, r=1, k=1, \gamma=1$ and $\vartheta=1$ fulfill the condition of theorem 1, then the following result holds:

$$S(\mu_1,\mu_2,\mu_3;z;\rho_1,\rho_2,\rho_3)$$

$$= \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2} \left[D_{x-y}^{-\rho_2} \{ \tau^{\rho_1-1} E_\theta(y+t) \} \right] (x-y)$$

$$= \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2} \left(D_{x-y}^{-\rho_2} \left\{ t^{\rho_1-1} {}_1\Psi_1 \left[\begin{matrix} (1,1) \\ (1,\theta) \end{matrix}; (y+t) \right] \right\} \right) (x-y)$$

$$= \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)} (x-y)^{1-\rho_1-\rho_2}$$

$$\times \left(D_{x-y}^{-\rho_2} \left\{ \tau^{\rho_1-1} H_{1,2}^{1,1} \left[\begin{matrix} (0,1) \\ (0,1)\ (0,\theta) \end{matrix}; -(y+t) \right] \right\} \right) (x-y)$$

Which is well known result by Gurjar M.K.(2020).

Theorem 4: Let $p,k,s \in \mathbb{N}, \theta, \vartheta, \rho \in \mathbb{C}$ with $\Re(\theta)>0, \Re(\vartheta)>0, \Re(\rho)>0$, then the double Dirichlet average of (p,s,k) Mittag-Leffler function $pE_{k,\theta,\vartheta}^{\gamma,s}(uozov)$ with the fractional derivative for $(k=x=2)$ is

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2) = \frac{(\mu_1)_n}{(\mu_1+\mu_2)_n} \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}$$

$$(x-y)^{1-\rho_1-\rho_2} \left[D_{x-y}^{-\rho_2} \{ \tau^{\rho_1-1} pE_{k,\theta,\vartheta}^{\gamma,s}(y+t) \} \right] (x-y)$$

Proof: Let take the double Dirichlet average for (k=x=2) of (p, s, k) Mittag-Leffler function $pE^{\rho,s}_{k,\theta,\vartheta}(u o z o v)$ into consideration with the help of above equation we get

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \int_0^1 \int_0^1 \{pE^{\gamma,s}_{k,\theta,\vartheta}(u \circ z \circ v)\} \mathrm{d}m_{\mu_1\mu_2}(\mathrm{u})\mathrm{d}m_{\rho_1\rho_2}(\mathrm{v}) \tag{4.1}$$

We get the following equation interpreting on equation (4.1) and interchanging the order of integration and summation

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \sum_{n=0}^{\infty} \frac{p^{(\gamma)}{}_{n,k,s}}{p^{\Gamma_{s,k}(n\theta+\vartheta)}} \frac{z^n}{n!} \int_0^1 \int_0^1 (u \circ z \circ v)^n \ \mathrm{d}m_{\mu_1\mu_2}(\mathrm{u}) \ \mathrm{d}m_{\rho_1\rho_2}(\mathrm{v})$$

Applying above equations , we have

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} \sum_{n=0}^{\infty} \frac{p^{(\gamma)}{}_{n,k,s}}{p^{\Gamma_{s,k}(n\theta+\vartheta)}}$$

$$\times \int_0^1 \int_0^1 [uv(a-b-c+d) + u(b-d) + v(c-d) + d]^n$$

$$\times\, u^{\mu_1-1}(1-u)^{\mu_2-1}\mathrm{du}\ \ v^{\rho_1-1}\,(1-v)^{\rho_2-1}\,\mathrm{dv}$$

We suppose that a=x, b=y, c=d=0 then we get

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} \sum_{n=0}^{\infty} \frac{p^{(\gamma)}{}_{n,k,s}}{p^{\Gamma_{s,k}(n\theta+\vartheta)}}$$

$$\times \int_0^1 \int_0^1 [uv(x-y) + uy]^n$$

$$\times\, u^{\mu_1-1}(1-u)^{\mu_2-1}\,\mathrm{du}\ v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{dv}$$

$$S(\mu_1, \mu_2; z; \rho_1, \rho_2) = \frac{\Gamma(\mu_1+\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)} \frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)} \sum_{n=0}^{\infty} \frac{p^{(\gamma)}{}_{n,k,s}}{p^{\Gamma_{s,k}(n\theta+\vartheta)}}$$

$$\times \int_0^1 \int_0^1 [vx + y(1-v)]^n$$

$$\times\, u^{\mu_1+n-1}(1-u)^{\mu_2-1}\ v^{\rho_1-1}(1-v)^{\rho_2-1}\,\mathrm{du}\,\mathrm{dv}$$

Also, with the help of well-known Beta and Gamma function, we get

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\mu_1+\mu_2)\Gamma(\rho_1+\rho_2)\Gamma(\mu_1+n)\Gamma(\mu_2)}{\Gamma(\mu_1)\Gamma(\mu_2)\Gamma(\rho_1)\Gamma(\rho_2)\Gamma(\mu_1+\mu_2+n)}\sum_{n=0}^{\infty}\frac{p^{(\gamma)}{}_{n,k,s}}{p^{\Gamma_{s,k}}(n\theta+\vartheta)}$$

$$\times\int_0^1[v(x-y)+y]^n\, v^{\rho_1-1}(1-v)^{\rho_2-1}\,dv$$

Suppose $v(x-y)=t$ then we have

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)\Gamma(\rho_2)}\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}\sum_{n=0}^{\infty}\frac{p^{(\gamma)}{}_{n,k,s}}{p^{\Gamma_{s,k}}(n\theta+\vartheta)}$$

$$\times\int_0^{x-y}(y+t)^n\left(\frac{t}{x-y}\right)^{\rho_1-1}\left(1-\frac{t}{x-y}\right)^{\rho_2-1}\frac{dt}{x-y}$$

Utilising the explanation of fractional calculus given in above result, we obtain

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)=\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}(x-y)^{1-\rho_1-\rho_2}$$

$$\times\left[D_{x-y}^{-\rho_2}\{\tau^{\rho_1-1}pE_{k,\theta,\vartheta}^{\gamma,s}(y+t)\}\right](x-y)$$

Special Cases:

1. Let $s=1, p=1$ and $k=1$ fulfill the condition of theorem 4.2, $k=1$ and $p=1$ then the following result holds:

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)$$

$$=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\{\tau^{\rho_1-1}E_{\theta,\vartheta}^{\gamma}(y+t)\}\right](x-y)$$

which is well known result by Gurjar M.K. (2020).

2. Let $s=1, p=1, k=1$ and $\gamma=1$ fulfill the condition of theorem 4.2, then the following result holds:

$$S(\mu_1,\mu_2;z;\rho_1,\rho_2)$$

$$=\frac{\Gamma(\rho_1+\rho_2)}{\Gamma(\rho_1)}\frac{(\mu_1)_n}{(\mu_1+\mu_2)_n}(x-y)^{1-\rho_1-\rho_2}\left[D_{x-y}^{-\rho_2}\{\tau^{\rho_1-1}E_{\theta,\vartheta}(y+t)\}\right](x-y)$$

which is well known result by Gurjar M.K. (2020).

Conclusion:

In this paper, we develop results for Dirichlet averaging associated with special functions generalized by fractional derivatives in the form of Bessel-Maitland and M-L functions. Dirichlet average of two and three variables, Dirichlet measure, partial derivatives. The relation between generalized special functions is obtained by fractional derivatives. Some results have been derived by Thakur, A.K. et. al. (2015), Gurjar, M.K. (2021) and are special cases from other main results.

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